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Introduction to Nijenhuis Geometry and its applications

Lecture 1

Sydney, Feb 2022

Plan of the Lecture 1

- Introduction
  - Historic introduction.
  - Definition of Nijenhuis operator.
  - Simplest examples and local description in the diagonalisable case
- Application in the theory of projectively equivalent metrics
  - How Nijenhuis operators appear in (metric) projective geometry.
  - Philosophy of applications and how it works in projective equivalence.
  - Proof of Levi-Civita Theorem 1896.
- Singular points of Nijenhuis operators.
  - Definition and first examples.
  - Why study of singular points is necessary?
- Application: topology of manifolds admitting strictly nonproportional projectively equivalent Riemannian metrics.
  - Singular points which may appear in the context of projectively equivalent Riemannian metrics.
  - Dimension 2.
  - Ideas leading to the proof in an arbitrary dimension.

Albert Nijenhuis

Albert Nijenhuis (November 21, 1926 – February 13, 2015),

Dutch-American mathematician who specialised in differential geometry and the theory of deformations in algebra and geometry, and later worked in combinatorics.

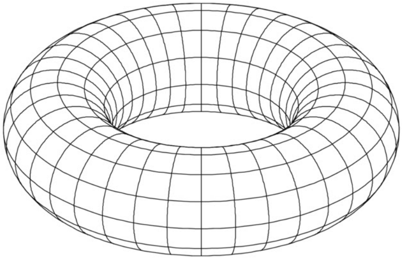
Alma mater: University of Amsterdam

Doctoral advisor: Prof. Jan Arnoldus Schouten

https://en.wikipedia.org/wiki/Albert_Nijenhuis

Branches of modern differential GEOMETRY

Space, manifold M^n $+$ Structure



Structure is usually defined by means of a tensor, like, g_{ij}, ω_{ij}, or p^{ij}

Naively, in coordinates, the geometric structure is defined by means of a matrix $A = \left(a_{ij}(x)\right)$ whose entries depend on coordinates $x = (x^1, \dots, x^n)$ and satisfy some algebraic and differential conditions.

Example: Symplectic geometry

Symplectic structure is defined by means of a differential form $\Omega = \left(\omega_{ij}(x)\right)$, tensor of type $(0,2)$ satisfying two algebraic and one differential conditions:

- skew symmetry $\quad \omega_{ij}(x) = -\omega_{ji}(x)$;
- non-degeneracy $\quad \det \Omega \neq 0$;
- closedness $\quad d\Omega = 0$, or $\frac{\partial \omega_{ij}}{\partial x^k} + \frac{\partial \omega_{jk}}{\partial x^i} + \frac{\partial \omega_{ki}}{\partial x^j} = 0$ for all i, j, k.

Equivalently, we may say that on the tangent space $T_x M$ of every point $x \in M$ we have a skew-symmetric non-degenerate bilinear (symplectic) form that is, in addition, closed.

Of course, the matrix $\Omega = \left(\omega_{ij}(x)\right)$ depends on the choice of local coordinates. Under a coordinate transform $(x^1, \dots, x^n) \mapsto (y^1, \dots, y^n)$, we have

$$\Omega \quad \mapsto \quad J^\top \Omega J$$

where $J = \left(\frac{\partial x^i}{\partial y^j}\right)$ is the Jacobi matrix of this transform.

Other examples

We have geometries defined by the following types of "matrices" (tensors)

symmetric bilinear forms on $T_x M$	$\mapsto$	Riemannian Geometry
skew-symmetric bilinear forms on $T_x M$	$\mapsto$	Symplectic Geometry
skew-symmetric bilinear forms on $T_x^* M$	$\mapsto$	Poisson Geometry
symmetric bilinear forms on $T_x^* M$	$\mapsto$	sub-Riemannian Geom.

Differential conditions.

- Vanishing of the Riemann curvature tensor (flat Riemannian geometry. Weaker conditions: e.g., Einstein condition):

$$\nabla_\xi \nabla_\eta - \nabla_\eta \nabla_\xi - \nabla_{[\xi,\eta]} = 0$$

- Closedness of Ω:

$$\xi\, \Omega(\eta, \zeta) + \Omega(\xi, [\eta, \zeta]) + (\text{cyclic permutation}) = 0$$

- Jacobi identity:

$$\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0$$

Nijenhuis geometry

Geometric structure is defined by means of a linear operator $L = \left(L^i_j(x)\right)$, tensor of type $(1,1)$, satisfying one differential conditions:

- Nijenhuis identity $\mathcal{N}_L = 0$:

$$\mathcal{N}_L(\xi, \nu) := L^2[\xi, \eta] - L[L\xi, \eta] - L[\xi, L\eta] + [L\xi, L\eta] = 0. \qquad (1)$$

Here ξ and η are arbitrary vector fields

Equivalently, on the tangent space T_xM of every point $x \in M$ we have an operator $L : T_xM \to T_xM$ with vanishing Nijenhuis torsion. Such an operator (tensor field of type (1,1)) is called a Nijenhuis operator. The left hand side of (1) is called the Niejnhuis torsion of L and denoted by $\mathcal{N}_L(\xi, \eta)$.

The matrix $L = \left(L^i_j(x)\right)$ depends on the choice of local coordinates. Under a coordinate transform $(x^1, \dots, x^n) \mapsto (y^1, \dots, y^n)$, we have

$$L \quad \mapsto \quad J^{-1}\, L\, J,$$

where $J = \left(\frac{\partial x^i}{\partial y^j}\right)$ as before.

Nijenhuis Geometry studies manifolds equipped with Nijenhuis operators. No other structure is a priori assumed. Of course, in some applications results proved about Nijenhuis operators will be combined with other structures.

More about the Nijenhuis torsion $\mathcal{N}_L$.

Let $L = (L^i_j(x))$ be an operator (field of endomorphisms, tensor of type $(1,1)$) on a manifold M.

Definition. The Nijenhuis torsion of L is defined by

$$\mathcal{N}_L(\xi,\eta) = L^2[\xi,\eta] - L[L\xi,\eta] - L[\xi,L\eta] + [L\xi,L\eta], \qquad (2)$$

where ξ and η are arbitrary vector fields on M.

Proposition. $\mathcal{N}_L$ is a tensor of type $(1,2)$.

Notice that L is skew-symmetric w.r.t. ξ and η (i.e., w.r.t. lower indices). Equivalently, we may say that $\mathcal{N}_L$ is a vector valued 2-form.

Proof. We need to check that, in coordinates, $\big(\mathcal{N}_L(\xi,\eta)\big)^k = N^k_{ij}\xi^i\eta^j$.
In other words, there are no partial derivatives of ξ^i and η^j in the right hand side of (2), i.e., all of them cancel out.
Equivalently, we need to show bi-linearity over $C^\infty(M)$, that is,

$$\mathcal{N}_L(\xi, f_1\,\eta_1 + f_2\,\eta_2) = f_1\,\mathcal{N}_L(\xi,\eta_1) + f_2\,\mathcal{N}_L(\xi,\eta_2)$$

for any functions f_1, f_2, and similar for ξ.

Nijenhus torsion is a tensor of type $(1, 2)$ (proof)

For constant coefficients f_1 and f_2, this is obviously true. Thus, we only need to show that 'a function can be factored out', that is,

$$\mathcal{N}_L\left(\xi, f\eta\right) = f\mathcal{N}_L\left(\xi, \eta\right).$$

Notice that for each term of (2) separately, this condition fails! It holds true only for their linear combination with appropriately chosen signs.

Below we use the Leibnitz property of the Lie bracket:
$[\xi, f\eta] = f[\xi, \eta] + \xi(f)\eta$, where $\xi(f)$ denotes the directional derivative of f along ξ.

$$\mathcal{N}_L\left(\xi, f\eta\right) = L^2[\xi, f\eta] - L[L\xi, f\eta] - L[\xi, Lf\eta] + [L\xi, Lf\eta] =$$

$$L^2[\xi, f\eta] - L[L\xi, f\eta] - L[\xi, f\, L\eta] + [L\xi, f\, L\eta] =$$

$$L^2\Big(f[\xi, \eta] + \xi(f)\eta\Big) - L\Big(f[L\xi, \eta] + L\xi(f)\eta\Big) - L\Big(f[\xi, L\eta] + \xi(f)L\eta\Big) + \Big(f[L\xi, L\eta] + L\xi(f)L\eta\Big) =$$

$$= f(L^2[\xi, \eta] - L[L\xi, \eta] - L[\xi, L\eta] + [L\xi, L\eta])$$
$$+ \xi(f)L^2\eta - L\xi(f)L\eta - \xi(f)L^2\eta + L\xi(f)L\eta =$$

$$f(L^2(f[\xi, \eta] - L[L\xi, \eta] - L[\xi, L\eta] + [L\xi, L\eta]) = f\,\mathcal{N}_L(\xi, \eta),$$

as required.

Nijenhuis torsion in local coordinates

We have shown that $\mathcal{N}_L$ is a tensor of type $(1, 2)$.

Proposition. The component $(\mathcal{N}_L)^k_{ij}$ in a local coordinate system $x^1, \dots, x^n$ takes the following form:

$$(\mathcal{N}_L)^i_{jk} = L^s_j \frac{\partial L^i_k}{\partial x^s} - L^s_k \frac{\partial L^i_j}{\partial x^s} - L^i_s \frac{\partial L^s_k}{\partial x^j} + L^i_s \frac{\partial L^s_j}{\partial x^k}.$$

Let ∂_{x^j} and ∂_{x^k} be basis vector fields. Since $\mathcal{N}_L(\partial_{x^j}, \partial_{x^k}) = \sum_i (\mathcal{N}_L)^i_{jk} \partial_{x^i}$, we only need to compute $\mathcal{N}_L(\partial_{x^j}, \partial_{x^k})$.

Let us do it:

$$\mathcal{N}_L(\partial_{x^j},\partial_{x^k}) = L^2[\partial_{x^j},\partial_{x^k}] - L[L\partial_{x^j},\partial_{x^k}] - L[\partial_{x^j},L\partial_{x^k}] + [L\partial_{x^j},L\partial_{x^k}] =$$

we use $L\partial_{x^i} = L_i^k \partial_{x^k}$ (with summation over k) and $[\partial_{x^i}, f^k\partial_{x^k}] = \frac{\partial f^k}{\partial x^i}\partial_{x^k}$

$$-L\left[L_j^s\partial_{x^s},\partial_{x^k}\right] - L\left[\partial_{x^j},L_k^s\partial_{x^s}\right] + \left[L_j^s\partial_{x^s},L_k^i\partial_{x^i}\right] =$$
$$\frac{\partial L_j^s}{\partial_{x^k}} L\partial_{x^s} - \frac{\partial L_k^s}{\partial_{x^j}} L\partial_{x^s} + L_j^s\frac{\partial L_k^i}{\partial x^s}\partial_{x^i} - L_k^i\frac{\partial L_j^s}{\partial_{x^i}}\partial_{x^s} =$$

$$\left(\frac{\partial L_j^s}{\partial_{x^k}}L_s^i - \frac{\partial L_k^s}{\partial_{x^j}}L_s^i + L_j^s\frac{\partial L_k^i}{\partial x^s} - L_k^s\frac{\partial L_j^i}{\partial_{x^s}}\right)\partial_{x^i}$$

as was to be proved (in the last term we interchanged i and s).

Trivial examples of Nijenuis operators

Example 1. Constant operator $L(x) = \left(L^i{}_j\right) \in \mathbb{R}^{n\times n}$ is Nijenhuis.

Proof. In the formula

$$(\mathcal{N}_L)^i_{jk} = L^s_j \frac{\partial L^i_k}{\partial x^s} - L^s_k \frac{\partial L^i_j}{\partial x^s} - L^i_s \frac{\partial L^s_k}{\partial x^j} + L^i_s \frac{\partial L^s_j}{\partial x^k}.$$

we differentiate constants and get zero. □

Example 2. Let Id be the identity operator, $\mathrm{Id} = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}$

and $f : M \to \mathbb{R}$ be a (smooth) function.
Then, $f \cdot \mathrm{Id}$ is a Nijenhuis operator.
Proof. If we substitute $L = f \cdot \mathrm{Id}$ the formula

$$\mathcal{N}_L = L^2[\xi, \eta] - L[L\xi, \eta] - L[\xi, L\eta] + [L\xi, L\eta]$$

everything cancels similar to that in the proof that $\mathcal{N}_L$ is a tensor. □

First nontrivial example: a description of Nijenhuis operators with n different (real) eigenvalues

Theorem (Haantjes 1955/Nijenhuis 1951) Suppose L is a Nijenhuis operator on M^n such that at $p \in M^n$ it has n different real eigenvalues. Then, there exists a local coordinate system $(x^1, ..., x^n)$ near p such that

$$L = \mathrm{diag}(f_1(x^1), ..., f_n(x^n)) = \begin{pmatrix} f_1(x^1) & & \\ & \ddots & \\ & & f_n(x^n) \end{pmatrix}$$

where $f_i = f_i(x^i)$ are functions of one (indicated) variable.

Remarks.

- In Lecture 2 we will deal also with complex-valued eigenvalues. Spoiler: no essential difference.
- The functions f_i have a clear geometric meaning: they are eigenvalues of L. If every of them has nonvanishing differential, they are functionally independent so one can take them as coordinates. In this case, $L = \mathrm{diag}(x^1, ..., x^n)$.

Further remarks.

- The case when L is semi-simple (=diagonalisable) in a neighbourhood of p is similar (is also due to Haantjes) and the proof below can be easily generalised to that case. In this case, L is blockdiagonal:

$$L = \mathrm{diag}(f_1(X_1)\mathsf{Id}_{m_1}, ..., f_k(X_k)\mathsf{Id}_{m_k}).$$

  Here k is the number of eigenvalues, $m_1, ..., m_k$ are their multiplicities, and X's are blocks of the coordinates: $X_1 = x^1, ..., x^{m_1}$, $X_2 = x^{m_1+1}, ..., x^{m_1+m_2}$,... .
- The difficulties appear when L has Jordan blocks, we will cover this case in further lectures.
- Theorem is "if and only if", the operators $L = \mathrm{diag}(f_1(x^1), ..., f_n(x^n))$ and $L = \mathrm{diag}(f_1(X_1)\mathsf{Id}_{m_1}, ..., f_k(X_k)\mathsf{Id}_{m_k})$ are Nijenhuis.

Proof of Haantjes Theorem

1. Step: We show the existence of the coordinates $x^1, ..., x^n$ such that in these coordinates L is diagonal, $L = \mathrm{diag}(f_1, ..., f_n)$ with $f_i = f_i(x^1, ..., x^n)$. In other words, the vectors ∂_{x^i} should be eigenvectors of L.

Let ξ and η be two eigenvectors of L, $L\xi = f\xi$ and $L\eta = g\eta$ with some functions f, g. We show that the commutator $[\xi, \eta]$ is a linear combination of ξ and η.

We use Nijenhuis condition:

$$
\begin{aligned}
0 &= L^2[\xi,\eta] - L[L\xi,\eta] - L[\xi, L\eta] + [L\xi, L\eta] = \\
&= L^2[\xi,\eta] - L[f\xi,\eta] - L[\xi, g\eta] + [f\xi, g\eta] = \\
&= L(L - f\mathrm{Id})[\xi,\eta] - (L - f\mathrm{Id})[\xi, g\eta] + \text{ lin. comb. of } \xi,\eta \\
&= (L - g\mathrm{Id})(L - f\mathrm{Id})[\xi,\eta] + \text{ lin. comb. of } \xi,\eta
\end{aligned}
$$

implying that $[\xi, \eta]$ is equal to a linear combination of ξ, η.

Then, the eigenvector fields are simultaneously integrable by the Frobenius Theorem, so there exists a coordinate system $x^1, ..., x^n$ such that in these coordinates $L = \mathrm{diag}(f_1, ..., f_n)$ with $f_i = f_i(x^1, ..., x^n)$.

Second step: we need to show that $f_i = f_i(x^i)$.

We use the coordinate formula

$$(\mathcal{N}_L)^i_{jk} = L^s_j \frac{\partial L^i_k}{\partial x^s} - L^s_k \frac{\partial L^i_j}{\partial x^s} - L^i_s \frac{\partial L^s_k}{\partial x^j} + L^i_s \frac{\partial L^s_j}{\partial x^k}.$$

Since our L is diagonal, the terms L^s_j with $s \neq j$ are zero, so the formula is reduced to (we assume $i \neq j$):

$$0 = (\mathcal{N}_L)^i_{ji} = \underbrace{L^j_j}_{f_j} \frac{\partial L^i_i}{\partial x^j} - 0 - \underbrace{L^i_i}_{f_i} \frac{\partial L^i_i}{\partial x^j} + 0 \ = (f_j - f_i) \frac{\partial L^i_i}{\partial x^j}$$

implying that f_i does not depend on the coordinate x^j as we claimed. □

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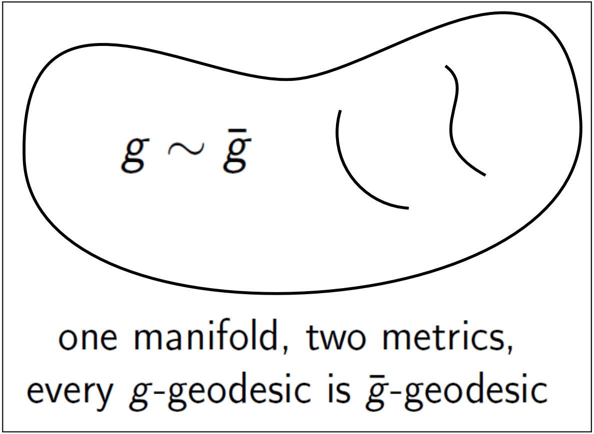
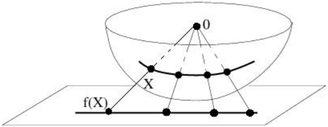
How Nijenhuis operators appears in the theory of projectively equivalent metrics

Two metrics (on one manifold) are projectively equivalent if they have the same unparametrized geodesics.



It is a natural topic started by Lagrange, Beltrami and Levi-Civita.

Examples of projectively equivalent metrics



Lagrange example 1789: Radial projection $f : S^2 \to \mathbb{R}^2$ takes geodesics of the sphere to geodesics of the plane, because geodesics on sphere/plane are intersection of planes containing 0 with the sphere/plane.

The example of Lagrange survives for all signatures and for all dimensions.

For a given n-dimensioanal metric g of constant curvature, the space of metrics projectively equivalent to g has dimension $\frac{(n+1)(n+2)}{2}$. The space of flat metrics projectively equivalent to g has dimension one less.

For example, in $n = 2$, the space of all projectively equivalent metrics is 6-dimensional, and the space of flat projectively equivalent metrics is 5-dimensional.

The equation for projective equivalence

- Of course projective equivalence can be written as a system of PDEs of the first order on the components of the metrics g and $\bar{g}$. The best way to write it down is as follows (Sinjukov 1966/Bolsinov-Matveev 2003):
- $$\nabla^g_k L_{ij} = \lambda_i g_{jk} + \lambda_j g_{ik},$$

  where
  $$L_{ij} = \left|\frac{\det \bar{g}}{\det g}\right|^{\frac{1}{n+1}} \bar{g}^{rs} g_{rj} g_{si} = \left|\frac{\det \bar{g}}{\det g}\right|^{\frac{1}{n+1}} g \bar{g}^{-1} g$$
  and $\lambda_i = \frac{1}{2} d \mathrm{trace}_g\, L$

  Convention. Above we used g for covariant differentiation ∇^g, and below we use g for index manipulations.

- Many geometers viewed this system as a system for L with given g. Let us now view this system as the system for g with a given OPERATOR $L = L^i_j$. We will see that it is linear in g:

$$\nabla^g_k L_{ij} = \underbrace{\lambda_i g_{jk} + \lambda_j g_{ik}}_{\text{already linear in } g \text{ with coefficients from trace } L}.$$

Let us consider the left hand side:

$$\nabla^g_k L_{ij} = \frac{\partial}{\partial x^k} L_{ij} - \Gamma^s_{ik} L_{sj} - \Gamma^s_{jk} L_{si} =$$

$$L^s_j \frac{\partial}{\partial x^k} g_{is} + g_{is} \frac{\partial}{\partial x^k} L^s_j - \Gamma_{sik} L^s_j - \Gamma_{sjk} L^s_i.$$

Since the Christoffel symbols of the first kind

$$\Gamma_{ijk} := \frac{1}{2}\left(\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^i}\right)$$

are evidently linear in g we see that the projective equivalence equations $\nabla^g_k L_{ij} = \lambda_i g_{jk} + \lambda_j g_{ik}$, is a linear system of geometric PDEs on g whose components are constructed by L^i_j.

Fact (Bolsinov-Matveev 2003). Compatibility conditions for this system are $\mathcal{N}_L = 0$.

Why one expected that the Nijenhuis condition should appear?

- The system on g (for a given L) is linear in g and is overdetermined $\left(\frac{(n+1)n^2}{2} - n \text{ equations on } \frac{n(n+1)}{2} \text{ unknowns}\right)$. The compatibility conditions are therefore nonempty and are quadratic in coefficients.
- The systems is geometric so compatibility conditions are tensorial.
- **Theorem (Puninskii 2014; follows from Kolar-Michor-Slovak 1993)** Every nontrivial differential-geometric operation from $(1,1)$-tensors to $(1,2)$-tensors that is homogeneous of degree 2 is the correspondence $L \mapsto \mathrm{const} \cdot \mathcal{N}_L$.
- **Remark.** "Trivial" operations of this type are just suitable algebraic expressions in L, $d\,\mathrm{trace}\, L$ and $d\,\mathrm{trace}\, L^2$.
- Thus, one expects that the condition $\mathcal{N}_L = 0$ should appear.

How Nijenhuis geometry helps for projectively equivalent metrics: proof of Levi-Civita theorem

(Tullio Levi-Civita, picture from Wikipedia)

- Levi-Civita is a famous Italian geometer; torsion-free connection compatible with a metric carries his name.
- In his doctoral dissertation (written under supervision of Ricci, 1896) he described all pairs of projectively equivalent Riemannian metrics.
- I will repeat the description and give a proof based on results in Nijenhuis geometry discussed today.

A frameworks for this and other similar applications: work in a coordinate system adapted to L

- Suppose we have a (geometric) system of PDEs whose coefficient are constructed by a Nijenhuis operator L.

- Work in a coordinate system such that L has the most simple form.

- In our case, the system is

$$g_{is}\frac{\partial}{\partial x^k}L^s_j + g_{js}\frac{\partial}{\partial x^k}L^s_i - \left(\frac{\partial g_{sk}}{\partial x^i} - \frac{\partial g_{ik}}{\partial x^s}\right)L^s_j - \left(\frac{\partial g_{sk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^s}\right)L^s_i = \frac{\partial \operatorname{trace}(L)}{\partial x^i}g_{jk} + \frac{\partial \operatorname{trace}(L)}{\partial x^j}g_{ik}. \tag{3}$$

- In the first part of the lecture we proved that (under some nondegeneracy assumptions) $L = \mathrm{diag}(x^1, ..., x^n)$. Since L is g-selfadjoint, also g is diagonal, $g = \mathrm{diag}(g_{11}, g_{22}, ..., g_{nn})$.
- If $i \neq j \neq k \neq i$, the equation (3) is trivially fulfilled. Also if $i = j \neq k$, the equation is trivially fulfilled.
- In the only remaining case, $j \neq i = k$, the equation reads

$$0 + 0 - \left(0 - \frac{\partial g_{ii}}{\partial x^j}\right)x^j - \left(\frac{\partial g_{ii}}{\partial x^j} - 0\right)x^i = 0 + g_{ii};$$

  i.e., $(x^j - x^i)\frac{\partial g_{ii}}{\partial x^j} = g_{ii}$. which is equivalent to $\frac{\partial}{\partial x^j}\left(\frac{g_{ii}}{x^j - x^i}\right) = 0$

  Then, $\frac{1}{x^j - x^i}g_{ii}$ does not depend on x^j implying $g_{ii} = \alpha(x^i)\prod_{j\neq i}(x^i - x^j)$.
  Thus, $g = \sum_i \alpha_i(x^i)\left(\prod_{j\neq i}(x^i - x^j)\right)(dx^i)^2$. This is the famous Levi-Civita Theorem 1896!

The form of the second metric

On the previous slide, we obtained the formula for the "first" metric g:
$g = \sum_i \alpha_i(x^i) \left(\prod_{j\neq i}(x^i - x^j) \right) (dx^i)^2$.
We also know that $L^i_j = \left| \frac{\det \bar{g}}{\det g} \right|^{\frac{1}{n+1}} \bar{g}^{is} g_{sj} = \mathrm{diag}(x^1, ..., x^n)$.

This implies that $\bar{g}$ is diagonal with

$$\bar{g}_{ii} = \left(\frac{1}{(x^i)^2} \prod_{j\neq i} \frac{x^i - x^j}{x_j} \right) (dx^i)^2.$$

Generally, the pair (g, L) contains full information about $\bar{g}$ and we rather work with (g, L).

Let us summarize and comment on remaining cases

- The assumption that g is Riemannian implies that L has no Jordan blocks. In the most nondegenerate case it has the (only possible) form $L = \mathrm{diag}(x^1, ..., x^n)$ in the coordinate system adapted to L. In this coordinate system, the equations for projective equivalence can be immediately solved.
- In the general case, $L = \mathrm{diag}(f_1(X_1)\mathrm{Id}_{m_1}, ..., f_k(X_k)\mathrm{Id}_{m_k})$. We again have a finite number of cases (determined by the dimension of the blocks) and the case can also be solved immediately.
- In the case of an arbitrary signature, the operator L may have Jordan blocks and complex eigenvalues. We will see that also in this case there is an explicit formula for L, and the system (3) can be solved.

- The same trick: work in coordinate system where L has the best form can be used in any such problem.

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Regular and singular points of Nijenhuis operators

- The manifold M^n and the Nijenhuis operator L are always assumed to be smooth, as smooth as we need for the proofs.
- Point $p \in M^n$ is algebraically generic for Nijenhuis operator L if the structure and number of Jordan blocks (the so-called Segre characteristic) does not change in some sufficiently small neighbourhood $U(p)$; the eigenvalues may depend on the point.

Example. We consider the Nijenhuis operator

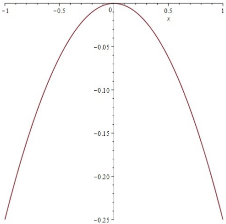
$$L = \mathrm{diag}(x^1, ..., x_n) = \begin{pmatrix} x^1 & & \\ & \ddots & \\ & & x^n \end{pmatrix}.$$

Singular points for it are the points where certain x^i equals certain x^j (with $i \neq j$).

A more complicated example

Let $n = 2$ and $L = \begin{pmatrix} x^1 & 1 \\ x^2 & 0 \end{pmatrix}$. The characteristic polynomial $\det(t\mathrm{Id} - L) = t^2 - x^1 t - x^2$.

We see that at the points where $\mathcal{D} := (x^1)^2 + 4x^2 \neq 0$ we have two different eigenvalues. Under the parabola $\{\mathcal{D} = 0\}$ the operator L has two complex-valued eigenvalues, above the parabola two real, and on the parabola a Jordan 2×2 block.

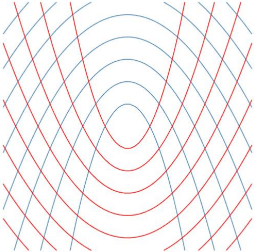


Two examples which will appear in our study of projectively equivalent metrics today

$$L = \begin{pmatrix} 0 & x^1 \\ x^1 & 2x^2 \end{pmatrix}$$

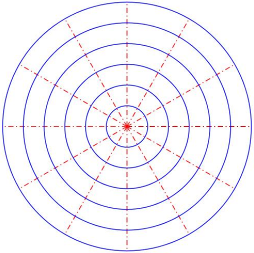
The matrix of L is symmetric so it has two real eigenvalues $x^2 \pm \sqrt{(x^1)^2 + (x^2)^2}$.

The contourlines of the eigenvalues are on the picture



$$L = \begin{pmatrix} (x^1)^2 & x^1x^2 \\ x^1x^2 & (x^2)^2 \end{pmatrix}$$

The matrix of L is symmetric so it has two real eigenvalues, 0 and $(x^1)^2 + (x^2)^2$.



Why to study singular points?

- **Informal motivations.**
  - In theory of differential equations, a half of modern papers are on singular points
  - In algebraic geometry, a half of papers are on singular points
  - In topology, many results were proved or can be reproved using functions with nondegenerate singularities (e.g. Morse functions)
- Motivation that will be discussed and demonstrated in the rest of this lecture: since on closed manifold we are expected to have singular points, so to get “global” results you need to path through singular points.

Where we are in our plan?

- Just done: Singular points of Nijenhuis operators.
  - Definition and first examples.
  - Why it is necessary to study singular points?
- Application: topology of manifolds admitting strictly nonproportional projectively equivalent Riemannian metrics.
  - Singular points which may appear in the context of projectively equivalent Riemannian metrics.
  - Dimension 2.
  - Ideas leading to the proof in arbitrary dimension.

Main theorem of this section

Theorem (Matveev 2006). Suppose M is a closed connected manifold with two projectively equivalent Riemannian metrics g and $\bar{g}$. Assume that there exists a point at which the operator $\bar{g}^{sj} g_{si}$ has n different eigenvalues. Then, the manifold can be covered by a product of spheres.

- I will explain the proof in the 2-dimensional case (Spoiler: singular points of Nijenhuis operators)
- I will shortly comment on the proof in the general case

The torus T^n has two projectively equivalent metrics

We slightly rewrite the formula for the metrics appearing in the Levi-Civita theorem: The form we obtained is:

$$g = \sum_i \alpha_i(x^i) \left(\prod_{j \neq i} (x^i - x^j) \right) (dx^i)^2, \quad L = \mathrm{diag}(x^1, ..., x^n)$$

and the form I will use is

$$g = \sum_i \alpha_i(x^i) \left(\prod_{j \neq i} (f_i(x^i) - f(x^j)) \right) (dx^i)^2, \quad L = \mathrm{diag}(f(x^1), ..., f(x^n)) \qquad (4)$$

(by already proven Haantjes Theorem, $L = \mathrm{diag}(f(x^1), ..., f(x^n))$ and our proof of Levi-Civita Theorem can be easily generalised to the case of this L).

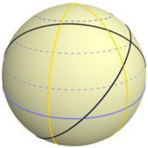
We use the formula (4) to construct projectively equivalent metrics on the torus: Take the standard periodic coordinates $x^1, ..., x^n \in S^1 \times \cdots \times S^1 := T^n$, functions $f_1(x^1), ..., f_n(x^n)$ (periodic with the period 2π, i.e., well-defined on S^1) with the property that $0 < f_i(x^i) < f_j(x^j)$ for $i \neq j$.

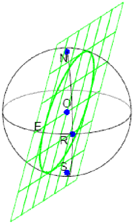
Then, the formula (4), provided $\alpha_i \neq 0$, gives a well-defined pseudo-Riemannian metric on the torus, and adjusting signs of α_i gives us a Riemannian metric on the torus. This metric admits a projectively equivalent metric.

Example on the sphere: Beltrami example 1865.

We consider the standard $S^n \subset R^{n+1}$ with the induced metric.

Fact. Geodesics of the sphere are the great circles, that are the intersections of the 2-planes containing the center of the sphere with the sphere.





Beltrami (1865) observed:

For every $A \in SL(n+1) \xrightarrow{\text{we construct}} a : S^n \to S^n, a(x) := \frac{A(x)}{|A(x)|}$

- a is a diffeomorphism
- a takes great circles (geodesics) to great circles (geodesics)
- a is an isometry iff $A \in O(n+1)$.

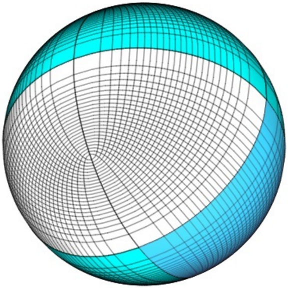
Thus, the pullback $a^*(g)$ is projectively equivalent to g, and is nonproportional to g if $A \notin \mathbb{R} \times O_{n+1}$

Haantjes-Levi-Civita coordinates for the Beltrami example

As we have seen in Haantjes Theorem, if Nijenhuis operator L has $n = \mathrm{dim} M$ different eigenvalues such that their differentials are not zero, then one can take them as coordinates in which $L = \mathrm{diag}(x^1, ..., x^n)$. In these coordinates the metric g is given by a relatively simple formula $g = \sum_i \alpha_i(x^i) \left(\prod_{j \neq i} (x^i - x^j) \right) (dx^i)^2$; the only freedom in the formula is the choice of the functions α_i.

Please, compare two pictures

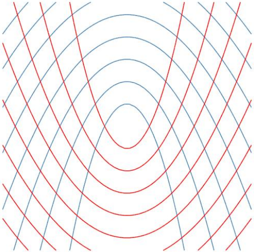
Example of Beltrami gives us projectively equivalent metrics and therefore L. On the picture we see the coordinate lines of these coordinates (which are sometimes called elliptic coordinates on the sphere). The picture corresponds to the generic case, when the symmetric matrix $A^T A$ (where A is from the Beltrami example) has 3 different eigenvalues



We clearly see the similarity of the picture with the 3rd picture of the singular point:

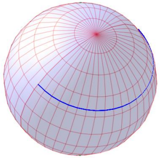
$$L = \begin{pmatrix} 0 & x^1 \\ x^1 & 2x^2 \end{pmatrix}$$

The matrix of L is symmetric so it has two real eigenvalues $x^2 \pm \sqrt{(x^1)^2 + (x^2)^2}$.
The contourlines of the eigenvalues are on the picture



Degenerate cases

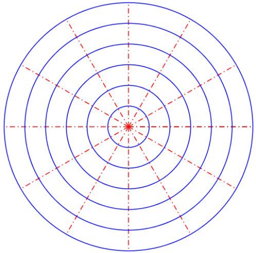
Example of Beltrami gives us projectively equivalent metrics and therefore L. On the picture we see the coordinate lines of these coordinates (which are sometimes called elliptic coordinates on the sphere). The picture corresponds to the degenerate case, when the symmetric matrix $A^T A$ has only 2 different eigenvalues



We clearly see the similarity with the 4th picture of the singular point:

$$L = \begin{pmatrix} (x^1)^2 & x^1 x^2 \\ x^1 x^2 & (x^2)^2 \end{pmatrix}$$

The matrix of L is symmetric so it has two real eigenvalues, 0 and $(x^1)^2 + (x^2)^2$.

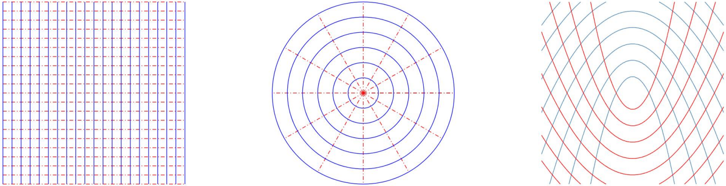


There is no other singularities.

Theorem (follows from Bolsinov-Matveev-Fomenko 1998). Consider two projectively equivalent metrics g, $\bar{g}$ in dimension $n = 2$ such that at least at one point they are nonproportional. Then, the corresponding Nijenhuis operator L, near every point, by a coordinate transformation and by addition of $\mathrm{const}\,\mathrm{Id}$, can be brought to one of the following forms discussed above:

$$L = \begin{pmatrix} f(x^1) & \\ & f_2(x^2) \end{pmatrix}, \ L = \begin{pmatrix} (x^1)^2 & x^1x^2 \\ x^1x^2 & (x^2)^2 \end{pmatrix} \text{ or } L = \begin{pmatrix} 0 & x^1 \\ x^1 & 2x^2 \end{pmatrix}.$$

In other words, the coordinate picture of Haantjes coordinates is as follows:



Corollary (Special case of Theorem announced above). In dimension 2, a closed manifold admitting projectively equivalent metrics which are nonproportional at least at one points is S^2, T^2, K^2 or RP^2.

Proof. Just try to glue another closed surface with these puzzle pieces (or observe that the index of the vector field generating red coordinate lines is always nonnegative).

Ideas of the proof in the general case

Theorem. Suppose M is a closed connected manifold with two projectively equivalent Riemannian metrics g and $\bar{g}$. Assume that there exists a point at which the operator $\bar{g}^{sj} g_{si}$ has n different eigenvalues. Then, the manifold can be covered by a product of spheres.

- First, we described singular points of the Nijenhuis operator $L(g, \bar{g})$. There are 4 cases. They are the same as those coming from the Beltrami example.
- Then, we observed (known before and was done by e.g. Solodovnikov 1956) that the (Levi-Civita) metric $g = \sum_i \alpha_i(x^i) \left(\prod_{j \neq i} (x^i - x^j) \right) (dx^i)^2$ has constant curvature if and only if $\alpha_i(x^i) = \frac{1}{P(x^i)}$ where P is a polynomial of degree $n+1$ which is the same for all i. This implies that if we know the values of α_i at one point we can extend them to any other point; this constructs a metric of constant curvature on the manifold implying the claim.

Conclusion and messages

- Nijenhuis operators appear in different subjects, especially in the "covariant" (=coordinate-independent) setups.
- A possible way to solve a problem where Nijenhuis operators appear is to work in a coordinate system adapted to a Nijenhuis operator.
- This method also can be used near singular points and in particular allows one to obtain global results.