Nijenhuis Geometry

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Lecture 10:
Singular points of scalar type
and linearization problem

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Singular points of scalar type

The Nijenhuis theorem might "break" at points, where the eigenvalues coincide.

Example: Consider the operator field

$$L = \begin{pmatrix} 0 & x \\ x & 2y \end{pmatrix}$$

The trace of L is $2y$ and determinant of L is $-x^2$. It is Nijenhuis by condition from Lecture 4

$$\begin{pmatrix} -2x & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2 \end{pmatrix} \begin{pmatrix} 2y & -x \\ -x & 0 \end{pmatrix}$$

The eigenvalues of this operator are

$$\lambda_{1,2} = y \pm \sqrt{x^2 + y^2}.$$

At the origin both eigenvalues coincide and everywhere else they are distinct. At the same time the eigenvalues are at most continuous at the origin. Thus, there is no smooth coordinate change, bringing operator field to diagonal form.

Singular points of scalar type

Consider point p and assume that all the eigenvalues at this point are real (if not, we apply the Splitting Theorem, split the neighbourhood into real and complex part and consider only the real part). The characteristic polynomial at p takes the form

$$\chi_L(t)|_{\mathrm{p}} = (t - \lambda_1)^{s_1}(t - \lambda_2)^{s_2} \dots (t - \lambda_k)^{s_k},$$

where λ_i are pairwise different eigenvalues with multiplicities s_i.

Applying again the Splitting Theorem, we see that neighbourhood splits into the direct product of k discs of dimensions s_i each. In each disc the point p is the point, where the eigenvalues of the operator coincide.

The singular point p is called **point of scalar type** if L is $\lambda\,\mathrm{Id}$ at this point.

Singular points of scalar type

Consider operator field L with property $L = \lambda\,\mathrm{Id}$ at point p. We get the following lemma.

Lemma

In a neighbourhood of point p with coordinates $x^1, \dots, x^n$, centered at this point, consider coordinate change $x(y)$, such that new coordinate system $y^1, \dots, y^n$ is centered at point p. The components $\frac{\partial L_i^k}{\partial x^j}\big|_{\mathrm{p}}$ are transformed as components of $(1, 2)$ tensor on tangent space $T_{\mathrm{p}}\mathrm{M}$

Proof: Under the coordinate change $x(y)$ the operator field is transformed as

$$\bar{L}_i^k = L_\beta^\alpha(x(y)) \frac{\partial x^\beta}{\partial y^i} \frac{\partial y^k}{\partial x^\alpha}.$$

We get

Singular points of scalar type

$$\frac{\partial \bar{L}_i^k}{\partial y^j} = \frac{\partial L_\beta^\alpha}{\partial x^\gamma}\frac{\partial x^\gamma}{\partial y^j}\frac{\partial x^\beta}{\partial y^i}\frac{\partial y^k}{\partial x^\alpha} + L_\beta^\alpha(x(y))\frac{\partial^2 x^\beta}{\partial y^j \partial y^i}\frac{\partial y^k}{\partial x^\alpha} + L_\beta^\alpha(x(y))\frac{\partial x^\beta}{\partial y^i}\frac{\partial}{\partial y^j}\left[\frac{\partial y^k}{\partial x^\alpha}\right]$$

Recall that

$$0 = \frac{\partial}{\partial y^j}\left[\delta_q^k\right] = \frac{\partial}{\partial y^j}\left[\frac{\partial y^k}{\partial x^\alpha}\frac{\partial x^\alpha}{\partial y^p}\right] = \frac{\partial}{\partial y^j}\left[\frac{\partial y^k}{\partial x^\alpha}\right]\frac{\partial x^\alpha}{\partial y^p} + \frac{\partial y^k}{\partial x^\alpha}\frac{\partial^2 x^\alpha}{\partial y^j \partial y^p}$$

This yields formula

$$\frac{\partial}{\partial y^j}\left[\frac{\partial y^k}{\partial x^\alpha}\right] = -\frac{\partial^2 x^\beta}{\partial y^j \partial y^p}\frac{\partial y^k}{\partial x^\beta}\frac{\partial y^p}{\partial x^\alpha}$$

Substituting it into the formula for derivative and taking all the formulas at p, we get that blue part is

$$\lambda\frac{\partial^2 x^\beta}{\partial y^j \partial y^i}\frac{\partial y^k}{\partial x^\beta} - \lambda\frac{\partial^2 x^\gamma}{\partial y^j \partial y^p}\frac{\partial y^k}{\partial x^\gamma}\frac{\partial y^p}{\partial x^\alpha}\frac{\partial x^\alpha}{\partial y^i} = 0.$$

The lemma is proved

Singular points of scalar type

The tensor $a_{ij}^k = \frac{\partial L_i^k}{\partial x^j}|_{\mathrm{p}}$ of type $(1, 2)$ on a linear space with fixed basis $\eta_1, \dots, \eta_n$ defines the structure of an algebra

$$\eta_i \star \eta_j = a_{ij}^k \eta_k.$$

Consider another definition of the algebra, which does not use the coordinates.

1. Take $\xi_0, \eta_0 \in T_{\mathrm{p}}\mathrm{M}$
2. Continue these vectors as vector fields on the entire neighbourhood of p as ξ, η. The continuation is arbitrary and smooth
3. Consider the following operation

$$\xi_0 \star \eta_0 = \big([\eta, L\xi] - \lambda[\eta, \xi]\big)|_{\mathrm{p}}$$

Lemma

The definition of $\star$ does not depend on continuation of ξ_0, η_0. In given coordinates and associated basis in $T_{\mathrm{p}}\mathrm{M}$ the structure constants of this operation are exactly $a_{ij}^k = \frac{\partial L_i^k}{\partial x^j}|_{\mathrm{p}}$.

Singular points of scalar type

Take decomposition of vector fields as

$$\xi = \xi_0 + \xi_1, \quad \eta = \eta_0 + \eta_1,$$

where ξ_1, η_1 vanish at the origin. We have

$$\begin{aligned}
&[\eta, L\xi]^k - \lambda[\eta, \xi]^k = [\eta, (L - \lambda\,\mathrm{Id})\xi]^k = [\eta_0 + \eta_1, L_\lambda \xi] = \\
&= [\eta_0, L_\lambda \xi_0] + [\eta_1, L_\lambda \xi] + [\eta_0, L_\lambda \xi_1] = \\
&= [\eta_0, L_\lambda \xi_0] + [\eta_1, L_\lambda \xi] + \left(\mathcal{L}_{\eta_0} L_\lambda\right)\xi_1 + L_\lambda[\eta_0, \xi_1]
\end{aligned}$$

Now note that if both ξ, η vanish at point, then $[\xi, \eta]$ also vanishes at the point. Together with fact that $L_\lambda = 0$ at p this yields that red part vanishes at point and the rest gives us the formula we need:

$$\left([\eta, L\xi]^k - \lambda[\eta, \xi]^k\right)|_{\mathrm{p}} = \frac{\partial L_i^k}{\partial x^j}|_{\mathrm{p}} \xi_0^i \eta_0^j$$

Left-symmetric algebras

Lemma

If L is Nijenhuis operator and p its singular point of scalar type, then the algebra, defined on $T_{\mathrm{p}}\mathrm{M}$ is left symmetric.

Proof: Actually this statement can be obtained by differentiating of $\mathcal{N}_L$ at p. But for the sake of later linearization we prove this statement in category of power series.

If in local coordinates the entries of operator field L are homogeneous polynomials of degree k and the entries of operator field R are homogeneous polynomials of degree m, then the entries of $[[L, R]]$ are homogeneous polynomials of degree $k + m - 1$.

Taking decomposition $L = L_1 + \dots$, where $\dots$ stands for the terms of higher order, we get

$$0 = [[L, L]] = [[L_1, L_1]] + \text{terms of order} > 1$$

Thus L_1 is Nijenhuis and it is exactly the right action operator of corresponding algebra.

Linearization problem

We call the left-symmetric algebra (LSA) on $T_{\mathrm{p}}\mathrm{M}$ **the isotropy LSA**. In local coordinates

$$(L_1)^k_i(x) = \frac{\partial L^k_i}{\partial x^\alpha} x^\alpha$$

Consider Nijenhuis operator L in the neighbourhood of the singular point p of scalar type. Denote L_1 to be its linear part.

The linearization problem is formulated as follows: when there exists a coordinate change $x(y)$, such that

$$L_1(y) = \left(\frac{\partial x}{\partial y}\right)^{-1} L(x(y)) \left(\frac{\partial x}{\partial y}\right)$$

The similar problem is well-known for vector fields. This problem is usually considered in three categories: formal, analytic and smooth.

Linearization of vector fields for $n = 2$

Formal category is the category of formal power series. Consider a vector field

$$\xi = \xi_1 + \xi_k + \dots,$$

where ξ_i is a vector field with homogeneous entries of order i. We assume that $\xi_1 = (x, \alpha y)$.

The formal linearization procedure is a sequence of coordinate changes that kill all the terms except ξ_1. Consider coordinate change $\bar{x} = x + f(x, y), \bar{y} = y + g(x, y)$, where f, g are homogeneous polynomials of degree k. We get

$$\mathcal{L}_\xi(\bar{x}) = x + \left(\xi_k^1 + \mathcal{L}_{\xi_1}(f)\right) + \dots,$$
$$\mathcal{L}_\xi(\bar{y}) = \alpha y + \left(\xi_k^2 + \mathcal{L}_{\xi_1}(g)\right) + \dots,$$

The condition of vanishing of terms of order k is

$$\xi_k^1 + \mathcal{L}_{\xi_1}(f) = f, \quad \xi_k^2 + \mathcal{L}_{\xi_1}(g) = \alpha g.$$

Linearization of vector fields for $n = 2$

Vector fields ξ_1 acts on the monomials as

$$\mathcal{L}_{\xi_1}(x^m y^s) = (m + \alpha n) x^m y^s.$$

The formal equations can written as

$$\xi_k^1 = (\mathrm{Id} - \mathcal{L}_{\xi_1})(f), \quad \xi_k^2 = (\alpha\,\mathrm{Id} - \mathcal{L}_{\xi_1})(g)$$

They have solution for arbitrary left-hand side if $m + \alpha n - 1 \neq 0$ and $m + \alpha n - \alpha \neq 0$ for $m + n \geq 2$. The "forbidden" values of α in this case are

$$\alpha = s, \quad \alpha = \frac{1}{s}, \quad \alpha = -\frac{p}{q}, \quad s, p, q \in \mathbb{N}, s \geq 2$$

The first two are called resonant nodes, the third is called resonant saddle.

Linearization of vector fields for $n = 2$

If α is not one of the forbidden values, then the operators have an inverse. In this case we get

$$\xi(\bar{x}) = x + \left(\xi_k^1 + \xi_1(f)\right) + \dots = \bar{x} + \dots,$$
$$\xi(\bar{y}) = \alpha y + \left(\xi_k^2 + \xi_1(g)\right) + \dots = \alpha\bar{y} + \dots,$$

where $\dots$ stand for terms of order higher then k. Thus, substituting the inverse coordinate change $x(\bar{x}, \bar{y}), y(\bar{x}, \bar{y})$ we get

$$\bar{\xi} = \bar{\xi}_1 + \bar{\xi}_{k+1} + \dots$$

Here $\bar{\xi}$ stands for components of our vector field in new coordinates. Continuing this process we get the formal coordinate change that transforms vector field into the linear one.

Linearization in diagonal case

Theorem

Consider Nijenhuis operator L with decomposition $L = L_1 + L_k + \dots$ at singular point p of scalar type for $k \geq 2$. Assume that in given coordinates the linear part has the form

$$L_1 = \begin{pmatrix} x^1 & 0 & 0 & \dots & 0 \\ 0 & x^2 & 0 & \dots & 0 \\ 0 & 0 & x^3 & \dots & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & \dots & x^n \end{pmatrix}$$

Then L can be linearized in both formal and analytic category

Proof of theorem: step 1

Recall that we have

$$[[L, L]] = [[L_1, L_1]] + 2[[L_1, L_k]] + \dots$$

Thus, the Frolicher-Nijenhuis bracket of L_1 and L_k vanishes. Let us calculate bracket:

$$
\begin{aligned}
&[[L_1, M]](\partial_{x^i}, \partial_{x^j}) = \\
=&L_1[M\partial_{x^i}, \partial_{x^j}] + L_1[\partial_{x^i}, M\partial_{x^j}] + M[L_1\partial_{x^i}, \partial_{x^j}] + M[\partial_{x^i}, L_1\partial_{x^j}]- \\
-&[L_1\partial_{x^i}, M\partial_{x^j}] - [M\partial_{x^i}, L_1\partial_{x^j}] = \\
=&L_1[M^\alpha_i\partial_{x^\alpha}, \partial_{x^j}] + L_1[\partial_{x^i}, M^\alpha_j\partial_{x^\alpha}] + M[x^i\partial_{x^i}, \partial_{x^j}] + M[\partial_{x^i}, x^j\partial_{x^j}]- \\
-&[x^i\partial_{x^i}, M^\alpha_j\partial_{x^\alpha}] - [M^\alpha_i\partial_{x^\alpha}, x^j\partial_{x^j}] = \\
=&-\frac{\partial M^\alpha_i}{\partial x^j}x^\alpha\partial_{x^\alpha} + \frac{\partial M^\alpha_j}{\partial x^i}x^\alpha\partial_{x^\alpha} - x^i\frac{\partial M^\alpha_j}{\partial x^i}\partial_{x^\alpha} + M^i_j\partial_{x^i} + x^j\frac{\partial M^\alpha_i}{\partial x^j}\partial_{x^\alpha} - M^j_i\partial_{x^j} = \\
=&\Big(M^i_j + (x^j - x^i)\frac{\partial M^i_i}{\partial x^j}\Big)\partial_{x^i} + \dots
\end{aligned}
$$

Proof of theorem: step 2

We get that vanishing of Frolicher-Nijenhuis bracket yields

$$M^i_j = (x^i - x^j)\frac{\partial M^i_i}{\partial x^j}.$$

Consider coordinate change $\bar{x}^i = x^i + f^i$ for $f^i = (L_k)^i_i$. By definition we have

$$\bar{L} = \left(\mathrm{Id} + \frac{\partial f}{\partial x}\right)^{-1} L \left(\mathrm{Id} + \frac{\partial f}{\partial x}\right) = L_1 + \left(L_k + L_1\frac{\partial f}{\partial x} - \frac{\partial f}{\partial x}L_1\right) + \dots,$$

where $\frac{\partial f}{\partial x}$ stands for Jacobi matrix of functions f^i. By definition we have that

$$L_k + L_1\frac{\partial f}{\partial x} - \frac{\partial f}{\partial x}L_1 = \mathrm{diag}\{f^1, \dots, f^n\}$$

Proof of theorem: step 3

We get

$$L_1 + \left(L_2 + L_1\frac{\partial f}{\partial x} - \frac{\partial f}{\partial x}L_1\right) = L_1 + \operatorname{diag}\{f^1, \dots, f^2\} = L_1(\bar{x}).$$

Thus, this coordinate change kills of the terms of order k. Proceeding in a similar way we get a formal coordinate change that linearizes the corresponding operator. Now we proceed to the analytic part. Consider L and the roots of characteristic polynomial

$$\chi_L(t) = 0.$$

The coefficients of the polynomial are analytic functions of coordinates. Thus, the formal power series that define the linearizing coordinate change are, in fact, the formal solutions to this system.

Applying Artin's theorem we get that there exist an analytic solutions to the same system approximating the formal solution to some order. The analytic solutions that approximate the formal in first order, define the analytic coordinate change. The theorem is proved.

Linearization in smooth category

The point P is said to be **critical** for a function $f(x, y)$ on the plane with coordinates x, y if $\frac{\partial f}{\partial x}|_P = \frac{\partial f}{\partial y}|_P = 0$.

Theorem (Parametric Morse lemma)

Consider an smooth function $f(x, y)$ on real plane with critical point at the coordinate origin. Assume that

$$\frac{\partial^2 f}{\partial x^2}(0,0) = \gamma \neq 0, \quad \frac{\partial^2 f}{\partial x \partial y}(0,0) = 0.$$

There exists a smooth centered at the origin coordinate change in the form $\bar{x} = g(x, y), \bar{y} = y$, such that

$$f(\bar{x}, \bar{y}) = \operatorname{sgn}(\gamma)\, \bar{x}^2 + k(\bar{y}), \tag{1}$$

where h is smooth function of one variable with $k(0) = k'(0) = 0$ and $k''(0) = \frac{\partial^2 f}{\partial y^2}(0,0)$.

Proof of Morse lemma

As $\frac{\partial^2 f}{\partial x^2}(0,0) \neq 0$, then by the Implicit Function Theorem there exists a smooth curve $r(y)$ such that

$$\frac{\partial f}{\partial x}(r(y), y) \equiv 0, \quad r(0) = 0.$$

Function $f(x, y)$ can be written in the following integral form

$$f(x,y) = f(r(y), y) + \int_0^1 \frac{\mathrm{d}}{\mathrm{d}\, t} f\big(tx + (1-t)r(y), y\big)\mathrm{d}\, t =$$

$$= f(r(y), y) + \big(x - r(y)\big) \int_0^1 \frac{\partial f}{\partial x}\big(tx + (1-t)r(y), y\big)\mathrm{d}\, t$$

Proof of Morse lemma

We denote

$$\Phi(x,y)=\int_0^1 \frac{\partial f}{\partial x}\big(tx+(1-t)x(y),y\big)\mathrm{d}\,t$$

By definition of $r(y)$ we have $\Phi(r(y),y)=\frac{\partial f}{\partial x}(r(y),y)\equiv 0$. For Φ the following integral formula holds

$$\Phi(x,y)=\Phi(r(y),y)+\int_0^1 \frac{\mathrm{d}}{\mathrm{d}\,t}\Phi\big(tx+(1-t)r(y),y\big)=$$

$$=\big(x-r(y)\big)\int_0^1 \frac{\partial \Phi}{\partial x}\big(tx+(1-t)r(y),y\big)$$

Proof of Morse lemma

Denoting $F(x,y) = \int\limits_0^1 \frac{\partial \Phi}{\partial x}\big(tx + (1-t)r(y), y\big)$ and substituting the expression into the original integral formula for f we get

$$f(x,y) = f(r(y),y) + \big(x - r(y)\big)^2 F(x,y).$$

In this formula $\frac{\partial^2 f}{\partial x^2}(0,0) = 2F(0,0) = \gamma \neq 0$. Consider functions

$$\bar{x} = (x - r(y))\sqrt{|F(x,y)|} = g(x,y), \quad \bar{y} = y. \tag{2}$$

These are smooth functions in a neighbourhood of coordinate origin and $g(0,0) = 0$.

Proof of Morse lemma

The Jacobian of $\bar{x}, \bar{y}$ at the origin is $\sqrt{\frac{|\gamma|}{2}} \neq 0$, thus, (2) defines a coordinate change in a neighbourhood of coordinate origin. In new coordinates after renaming $f(r(\bar{y}), \bar{y}) = k(\bar{y})$ function f takes the form (1). By definition of curve $r(y)$ we get

$$k'(y) = \frac{\partial f}{\partial x}\Big(r(y), y\Big) r'(y) + \frac{\partial f}{\partial y}\Big(r(y), y\Big) = \frac{\partial f}{\partial y}\Big(r(y), y\Big),$$

$$k''(y) = \frac{\partial^2 f}{\partial x \partial y} r'(y) + \frac{\partial^2 f}{\partial y^2}$$

This implies that

$$k(0) = k'(0) = 0,$$

$$k''(0) = \frac{\partial^2 f}{\partial y^2}(0, 0).$$

Linearization in smooth case

Equipped with this lemma we can prove the following proposition.

Theorem

In dimension two consider Nijenhuis operator L around the singular point of scalar type p. Without loss of generality assume that $L = 0$ at p and linear part L_1 is

$$L_1 = \begin{pmatrix} y & x \\ x & y \end{pmatrix}$$

Then there exists a smooth coordinate change, centered at p that transforms L into its linear part.

Proof: Consider functions $f = \operatorname{tr} L$ and $g = \det L$. By definition the Taylor decompositions of these functions at point p have the form:

$$f = 2y + \dots,$$
$$g = y^2 - x^2 + \dots.$$

Here $\dots$ stand for terms of orders > 1 and > 2 respectively.

Proof of Theorem

As $\mathrm{d}\, f \neq 0$ at the origin, we can take it as $2y$. Thus, we have $f = 2y$.

Now we treat y as a parameter and x as a variable. We have that

$$\frac{\partial^2 g}{\partial x^2}(0,0) = -2 \neq 0, \quad \frac{\partial^2 g}{\partial x \partial y}(0,0) = 0, \quad \frac{\partial^2 g}{\partial y^2}(0,0) = 2$$

By parametric Morse lemma there exists a coordinate change that leaves y untouched and transforms the function into the

$$g = -x^2 + k(y)$$

with property

$$k''(0) = \frac{\partial^2 g}{\partial y^2}(0,0) = 2.$$

Proof of Theorem

Thus, we get that trace of L is $2y$ and determinant is $-x^2 + k(y)$. This imply that both functions are independent almost everywhere on a neighbourhood of p. Actually they are dependant only on the curve $x = 0$.

Using formulas from lectures 2 and 4 we can reconstruct the operator in the form

$$L = \frac{1}{f_x g_y - f_y g_x} \begin{pmatrix} g_y & -f_y \\ -g_x & f_x \end{pmatrix} \begin{pmatrix} f & 1 \\ g & 0 \end{pmatrix} \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}$$

Substituting the f, g we have into this formula, we get

$$L = \begin{pmatrix} \frac{1}{2}k' & \frac{k'(4y-k')+4x^2-4k}{4x} \\ x & 2y - \frac{1}{2}k' \end{pmatrix}$$

Proof of theorem

We rewrite this term as

$$\frac{k'(4y - k') + 4x^2 - 4k}{4x} = x + \frac{k'(4y - k') - 4k}{4x}$$

It is smooth if and only if

$$k'(4y - k') - 4k = 0$$

Differentiating both sides by y we get

$$\frac{1}{2}k''(y)(2y - k') = 0.$$

As $k''(0) = 2$, we have that $k' = 2y$. As $k(0) = 0$ we get $k(y) = y^2$. Thus, we see that the coordinates we have constructed by parametric Morse lemma are, in fact, linearizing coordinate change.

No linearization case

Consider Nijenhuis operator

$$L = \begin{pmatrix} y + yx^2 & x + x^3 \\ -xy^2 & y - yx^2 \end{pmatrix} = \begin{pmatrix} y & x \\ 0 & y \end{pmatrix} + \begin{pmatrix} yx^2 & x^3 \\ -xy^2 & -yx^2 \end{pmatrix}$$

Assume that it has linearizing coordinate change. In these coordinates trace is $2y$ and determinant is y^2. In particular, the discriminant of characteristic polynomial

$$D = (\operatorname{tr} L)^2 - 4 \det L = 4y^2 - 4y^2 = 0$$

identically vanish. At the same time for the perturbed operaor we have

$$(\operatorname{tr} L)^2 - 4 \det L = -4x^2y^2.$$

Thus, there are no linearization.