Nijenhuis Geometry

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Lecture 11:
Smooth vs analytic linearization
and non-degenerate left-symmetric algebras

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Linearization of vector fields: smooth case

Consider vector field ξ on a plane. We assume that coordinates x, y are fixed and that ξ has a **critical point** at the coordinate origin. We assume that it has decomposition $\xi = \xi_1 + \ldots$ and the linear part is

$$\xi_1 = (x, \alpha y)$$

for some $\alpha \neq 0$. The next is classical result about the linearization of vector fields.

Theorem

For $p, q, m \in \mathbb{N}, m \geq 2$ and a smooth vector field ξ as above we have that:

1. *If α is not $-\frac{p}{q}, m, \frac{1}{m}$, then the smooth linearization always exists*
2. *If α is either one of $-\frac{p}{q}, m, \frac{1}{m}$, then there exist vector fields ξ without smooth linearization*

One might recall that the forbidden values of α are exactly the forbidden values for formal linearization we have obtained in Lecture 10.

Linearization of vector fields: analytic case

The continuous fraction for α is a decomposition of α in the form

$$\alpha = q_0 + \frac{1}{q_1 + \frac{1}{q_2 + \ldots}},$$

where $q_0 \in \mathbb{Z}$ and $q_i, i \geq 1$ are in $\mathbb{N}$. The decomposition is finite if and only if α is rational.

Let $[q_0, q_1, q_2, \ldots]$ be a decomposition of an irrational α into the continuous fraction. If the series

$$B(x) = \sum_{i=0}^{\infty} \frac{\log q_{i+1}}{q_i}$$

converges, then α is a **Brjuno number**. Denote Σ to be the set of negative irrational numbers, that are not Brjuno numbers.

Linearization of vector fields: analytic case

Theorem

For $p, q, m \in \mathbb{N}, m \geq 2$ and analytic vector fields ξ we have that:

1. *If α is not in Σ or $-\frac{p}{q}, m, \frac{1}{m}$, then the analytic linearization always exists*
2. *If α is either one of $-\frac{p}{q}, m, \frac{1}{m}$ or in Σ, then there exist vector fields ξ without analytic linearization*

Note that in case when α is either of $-\frac{p}{q}, m, \frac{1}{m}$ even formal linearization might not exists (we have discussed this question in Lecture 10). The case $\alpha \in \Sigma$ is very complicated and existence of vector fields, that have formal linearization but no analytic (that is all their formal linearizations are divergent series) is a result by Yoccoz, one of the reasons he received the Fields medal.

Integrals of vector fields near critical points

For a given vector field ξ, function f is called an integral if $\mathcal{L}_\xi f = 0$. The existence of integrals around regular point is trivial, but around the critical points of ξ is not so much.

Example: Consider linear vector field $\xi = (\alpha x, y)$ for $\alpha < 0$. Fix the constant $s = -\frac{1}{\alpha} \in \mathbb{R}^+$. Define the function f as follows:

$$f(x,y) = \begin{cases} \exp\left(-\frac{1}{x^{2s}y^2}\right), & xy \neq 0 \\ 0, & xy = 0. \end{cases}$$

Defining all the partial derivatives of $f(x,y)$ as zero on the coordinate cross $xy = 0$ we obtain a function that is smooth on the entire plane and flat on the $xy = 0$. The partial derivatives of $f(x,y)$ satisfy the following identities:

$$\frac{\partial f}{\partial x}(x,y) = \frac{2s}{x^{2s+1}y^2} f(x,y), \quad \frac{\partial f}{\partial y}(x,y) = \frac{2}{x^{2s}y^3} f(x,y).$$

Integrals of vector fields near critical points

In particular

$$\alpha x f_x + y f_y = 0$$

and, thus, for $\alpha < 0$ function $f(x, y)$ is a smooth integral for the linear vector field $\xi = (x, \alpha y)$.

Example: Consider the same linear vector field $\xi = (x, \alpha y)$. Assume now that we have an analytic first integral $f = f_k + f_{k+1} + \dots$ (here f_k is first non-zero term). By definition

$$0 = \mathcal{L}_\xi f = \mathcal{L}_\xi(f_k) + \dots$$

Here $\dots$ stands for terms of order higher then k. In Lecture 10 we have shown that $\mathcal{L}_\xi$ acts on the space of monomials of degree k as a diagonal operator with eigenvalues $n + \alpha m$, where $m + n = k$. The kernel of $\mathcal{L}_\xi$ is non-empty if and only if $\alpha = -\frac{p}{q}$.

Integrals of vector fields near critical points

We get the following lemma

Lemma (Integrals of linear vector fields)

Consider linear vector field $\xi = (\alpha x, y)$ with $\alpha \neq 0$ in the neighbourhood of the origin. Then

1. *It has non-constant smooth first integral if and only if $\alpha < 0$*
2. *It has non-constant analytic first integral if and only if $\alpha = -\frac{p}{q}$ with $p, q \in \mathbb{N}$*

Proof: The analytic part we have actually proved in our second example. The first example proved the half of the first statement. Now we need to show that for $\alpha > 0$ there is no smooth intergal. The ODE $\dot{x} = \alpha x, \dot{y} = y$ can be explicitly integrated:

$$x(t) = c_1 \exp \alpha t, \quad y(t) = c_2 \exp t,$$

where c_1, c_2 are arbitrary constants. For $\alpha > 0$ we have $x(t), y(t) \to 0$ as $t \to -\infty$. The closures of all integral curves contain coordinate origin, thus, the integral is locally constant.

Integrals of vector fields near critical points

Corollary (Necessary condition for existing of integrals)

Consider smooth (analytic) vector field ξ with critical point at the origin and linear part $\xi_1 = (x, \alpha y)$ for $\alpha \neq 0$. Then if ξ has smooth (analytic) integral in the neighbourhood of the origin, then:

1. In smooth case $\alpha < 0$
2. In analytic case $\alpha = -\frac{p}{q}$

Proof: In analytic case we take the same decomposition of integral $f = f_k + \dots$ and get that f_k is an integral of ξ_1. Thus, by Lemma $\alpha = -\frac{p}{q}$.

In smooth case we need to show, that $\alpha > 0$ can not have any integrals. For almost all $\alpha > 0$ we can linearize vector field and get that the corresponding linear vector field has no integrals.

The case $\alpha = m, \frac{1}{m}$ requires the application of theory of normal forms (which we omit), which yield the same result.

Analytic vs smooth linearization

Recall that the left-symmetric algebra $\mathfrak{b}_{1,\alpha}$ is a left-symmetric algebra with corresponding linear Nijenhuis operator

$$\begin{pmatrix} 0 & x \\ 0 & \alpha y \end{pmatrix}$$

For $y \neq 0$ the operator has pairwise distinct eigenvalues. For $y = 0$ it is a Jordan block. One eigenvalue is constant. The corresponding left-symmetric algebra is associative and not commutative.

Analytic vs smooth linearization

Consider Nijenhuis operator L with point p of scalar type and $L = L_1 + \dots$ at this point (that is L vanishes at p). Assume that L_1 is in the form from the previous slide.

If linearization exists, then one of the eigenvalues of L is constant around the point p. But before attacking this problem, we ask a simpler question: when L has smooth (analytic) eigenvalues? We already know, that this is not always the case.

The smooth (analytic) eigenvalue

Lemma

Consider an analytic (smooth) function $f(x, y)$ on real plane with critical point at the coordinate origin. Assume, that $f(0,0) = 0$ and

$$f_x(0,0) = f_y(0,0) = f_{xx}(0,0) = f_{xy}(0,0) = f_{yy}(0,0) = 0$$

For every $\alpha \neq 0$ there exist analytic (smooth) functions $h(x, y)$ and $k(y)$ such, that

$$h(0,0) = 0, \quad \Big(h_y(0,0)\Big)^2 = \frac{\alpha^2}{4} \neq 0, \quad h_x(0,0) = 0,$$
$$k(0) = k'(0) = k''(0) = 0$$

and

$$f(x, y) = \frac{\alpha^2}{4} y^2 - h^2(x, y) + k(x). \tag{1}$$

The smooth (analytic) eigenvalue

Proof: To prove the Lemma we apply parametric Morse Lemma to $f - \frac{\alpha^2}{4}y^2$ and change the names of coordinates.

In our case the trace of L has linear part αy and $\det L = f(x, y)$ satisfies the conditions of lemma: it has no linear and quadratic part. Applying the lemma we get

$$f(x, y) = \frac{\alpha^2}{4}y^2 - h^2(x, y) + k(x). \tag{2}$$

For Nijenhuis operator L we get:

$$\begin{aligned} &L_1^1 L_2^2 - L_2^1 L_1^2 = f, \\ &L_1^2 = -\frac{1}{\alpha} f_x, \quad L_1^1 = \frac{1}{\alpha} f_y, \\ &L_1^1 + L_2^2 = \alpha y. \end{aligned} \tag{3}$$

We have applied the condition for operator to be Nijenhuis in dimension two.

The smooth (analytic) eigenvalue

This yields the following formula

$$\frac{1}{\alpha} f_y\left(\alpha y - \frac{1}{\alpha} f_y\right) + \frac{1}{\alpha} L_2^1 f_x - f = 0$$

Substitititing the formula (2) for f into it we get

$$\left(\frac{\alpha}{2} y - \frac{2}{\alpha} h h_y\right)\left(\frac{\alpha}{2} y + \frac{2}{\alpha} h h_y\right) - \frac{\alpha^2}{4} y^2 + h^2 - k + \frac{1}{\alpha} L_2^1\Big(-2 h h_x + k'\Big) = 0$$

As $h_y(0,0) \neq 0$, then by the Implicit Function Theorem in sufficiently small neighbourhood of the origin there exists an analytic (smooth) curve $s(x)$ such, that

$$h(x, s(x)) = 0, \quad s(0) = 0, \quad s'(0) = -h_x(0,0)\Big(h_y(0,0)\Big)^{-1} = 0.$$

Substituting it into the condition and renaming $L_2^1(x, s(x)) = r(x)$ we get

$$\frac{1}{\alpha} r(x) k' - k = 0.$$

The smooth (analytic) eigenvalue

Recall that L_2^1 at the origin has linear part x. We can rewrite the equation as

$$(\ln k)' = \frac{k'}{k} = \frac{\alpha}{r(x)} = \frac{\alpha}{x + x^2 q(x)} = \alpha \left(\frac{1}{x} - \frac{q(x)}{1 + xq(x)} \right).$$

Integrating and taking exponent, we get that $k(x) = cx^{\alpha} F(x)$ for some smooth (analytic) $F(x)$ that is not zero at the origin and constant c. We know that $k'(0) = k''(0) = 0$.

We have the following two possibilities:

1. $\alpha \in \mathbb{N}$ and $\alpha \geq 3$
2. $k \equiv 0$ otherwise

The smooth (analytic) eigenvalue: singular α

Example: Consider Nijenhuis opetator

$$L = \begin{pmatrix} 0 & x \\ x^{\alpha-1} & \alpha y \end{pmatrix}$$

For $\alpha \in \mathbb{N}$ and $\alpha \geq 3$ this is smooth (analytic) operator field. The eigenvalues of this field are

$$-\frac{\alpha}{2} y \pm \frac{1}{2}\sqrt{\alpha^2 y^2 + 4x^\alpha}.$$

They are not even C^1 at the origin. Thus, the first case yields bad eigenvalues.

Moreover, as $\det L = -x^\alpha$, then the corresponding operator field is not linearizable

The smooth (analytic) eigenvalue: non-singular α

In this case $k = 0$ and $f = \frac{\alpha^2}{4} - h^2$. We see that in this case the eigenvalues of the Nijenhuis operator are

$$\mu_{1,2} = \frac{\alpha}{2} y \pm h(x, y).$$

They are both smooth (analytic) functions. We have

$$\mathrm{d}\,\mu_1 = \frac{\alpha}{2}\mathrm{d}\,y + \mathrm{d}\,h, \quad \mathrm{d}\,\mu_2 = \frac{\alpha}{2}\mathrm{d}\,y - \mathrm{d}\,h.$$

At the origin $h_y(0, 0)$ is either $\frac{\alpha}{2}$ or $-\frac{\alpha}{2}$. Thus, the differential of at least one eigenvalue is non-zero at the origin.

Recall that eigenvalues satisfy the identity

$$L^* \mathrm{d}\,\mu = \mu \mathrm{d}\,\mu.$$

The smooth (analytic) eigenvalue: non-singular α

Taking coordinate change, we get that L is in the form

$$L = \begin{pmatrix} L_1^1 & L_2^1 \\ 0 & \alpha y \end{pmatrix}$$

Note that the coordinate change did not change the linear part. The trace is $\alpha y + L_1^1$ and determinant is $\alpha y L_1^1$. The vanishing of Nijenhuis torsion yields

$$L_2^1 \frac{\partial L_1^1}{\partial x} + \left(\alpha y - L_1^1\right) \frac{\partial L_1^1}{\partial y} = 0. \qquad (4)$$

In coordinates x, y the vector field $\xi = \left(L_2^1, \alpha y - L_1^1\right)$ has L_1^1 as its integral and the linear part ξ is $\xi_1 = (x, \alpha y)$.

We are ready to proceed to constant eigenvalues

The constant eigenvalue: smooth case

We start with an example.

Example: Denote $f(x,y)$ to be the smooth first integral of $\xi = (x, \alpha y)$ for $\alpha < 0$. Define smooth function $g(x,y)$:

$$g(x,y) = \begin{cases} \frac{x}{y} f(x,y) & xy \neq 0 \\ 0 & xy = 0. \end{cases}$$

For $xy \neq 0$ this function has the derivatives of all orders. We define all the derivatives of g to be zero on $xy = 0$. As a result we get a smooth function and by the definition of this function

$$g f_x = f f_y. \tag{5}$$

Consider operator field

$$L = \begin{pmatrix} f(x,y) & x + g(x,y) \\ 0 & \alpha y \end{pmatrix}.$$

The constant eigenvalue: smooth case

The trace is $\alpha y + f$ and determinant is $\alpha y f$. The vanishing of Nijenhuis torsion is

$$\begin{pmatrix} f_x, & f_y + \alpha \end{pmatrix} \begin{pmatrix} \alpha y & -x - g \\ 0 & f \end{pmatrix} = \begin{pmatrix} \alpha y f_x, & -x f_x - g f_x + f f_y + \alpha f \end{pmatrix} =$$

$$= \begin{pmatrix} \alpha y f_x, & \alpha y f_y + \alpha f \end{pmatrix} = \mathrm{d} \det L.$$

Here we have used $g f_x = f f_y$ and $x f_x + \alpha y f_y = 0$. Thus, the corresponding operator field is Nijenhuis and its determinant is not zero almost everywhere around p.

We see that for $\alpha < 0$ the smooth eigenvalues exist, but they are not necessary constant. In particular this implies that for these values of α does not necessary exists.

The constant eigenvalue: analytic case

Example: Consider $f = x^p y^q$. This is an integral of $\xi = (x, \alpha y)$ for $\alpha = -\frac{p}{q}$ and $p, q \in \mathbb{N}$. In this case the function $g(x, y)$ from the previous example is simply

$$g(x, y) = \frac{q}{p} x^{p+1} y^{q-1}$$

The identity

$$g f_x = q x^{2p} y^{2q-1} = f f_y$$

The corresponding operator field is

$$L = \begin{pmatrix} x^p y^q & x + \frac{q}{p} x^{p+1} y^{q-1} \\ 0 & -\frac{p}{q} y \end{pmatrix}.$$

We see that in both analytic and smooth case for α having integral there are no linearization.

Constant eigenvalue: smooth case

We have proved the following theorem.

Theorem (Smooth case)

Assume that smooth Nijenhuis operator L *at singular point of scalar type* p *has linear part*

$$L_1 = \begin{pmatrix} 0 & x \\ 0 & \alpha y \end{pmatrix}.$$

Then we get

1. *If* $\alpha = m \in \mathbb{N}$ *and* $m \geq 3$ *then* L *does not necessarily have smooth eigenvalues around* p *(and, therefore,* L *cannot be linearized)*
2. *If* $\alpha < 0$ *then eigenvalues of* L *are smooth, but both may not be constant (and, therefore,* L *cannot be linearized)*
3. *If* $\alpha > 0$ *and* $\alpha \neq m \in \mathbb{N}$ *for* $m \geq 3$*, then there exists a smooth coordinate change that transforms* L *into*

$$L = \begin{pmatrix} 0 & f(x, y) \\ 0 & \alpha y \end{pmatrix}$$

Constant eigenvalue: analytic case

Theorem (Analytic case)

Assume that analytic Nijenhuis operator L at singular point of scalar type p *has linear part*

$$L_1 = \begin{pmatrix} 0 & x \\ 0 & \alpha y \end{pmatrix}.$$

Then we get

1. *If $\alpha = m \in \mathbb{N}$ and $m \geq 3$ then L does not necessarily have analytic eigenvalues around p (and, therefore, L cannot be linearized)*
2. *If $\alpha = -\frac{p}{q}$ for $p, q \in \mathbb{N}$ then eigenvalues of L are analytic, but both may not be constant (and, therefore, L cannot be linearized)*
3. *If $\alpha \neq -\frac{p}{q}$ for $p, q \in \mathbb{N}$ and $\alpha \neq m \in \mathbb{N}$ for $m \geq 3$, then there exists an analytic coordinate change that transforms L into*

$$L = \begin{pmatrix} 0 & f(x, y) \\ 0 & \alpha y \end{pmatrix}$$

Linearization: the last step

We have arrived to the following question: when does the operator field in the form

$$L = \begin{pmatrix} 0 & f(x,y) \\ 0 & \alpha y \end{pmatrix} \tag{6}$$

can be linearized? First note that $\operatorname{tr} L = \alpha y$ and after the linearization it stays the same. Thus, we are looking for coordinate change in the form $\bar{x} = g(x,y), \bar{y} = y$. Calculating the coordinate transformation we get

$$R = \begin{pmatrix} g_x & g_y \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & f \\ 0 & \alpha y \end{pmatrix} \begin{pmatrix} \frac{1}{g_x} & -\frac{g_y}{g_x} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & fg_x + \alpha y g_y \\ 0 & \alpha y \end{pmatrix}$$

For the linearization the condition $fg_x + \alpha y g_y = g$ must hold. Denoting $\xi = (f(x,y), \alpha y)$, we get that $\mathcal{L}_\xi(g) = g$. This yields the following Corollary.

Corollary

The Nijenhuis operator field in form (6) is linearized if and only if vector field $\xi = (f(x,y), \alpha y)$ can be linearized using "triangular" coordinate change $\bar{x} = g(x,y), \bar{y} = y$

Linearization: the last step

Lemma

Assume that $\xi = (f(x,y), \alpha y)$ *with* $f(x,y)$ *having linear part* x *at the origin can be linearized. Then there exists a triangular linearization coordinate change.*

Proof: The linearization in our case means that $\mathcal{L}_\xi g = g$ and $\mathcal{L}_\xi f = \alpha f$. Calculating

$$\mathrm{d}\,(\mathcal{L}_\xi g) = \frac{\partial g}{\partial x^\alpha}\frac{\partial \xi_1^\alpha}{\partial x^\beta}\mathrm{d}\,x^\beta + \cdots = \mathrm{d}\,g$$

Substituting coordinate origin we get that everything denoted ... vanishes. We get that $\mathrm{d}\,g$ at the origin is an eigenvalue of the dual to operator $\frac{\partial \xi_1^\alpha}{\partial x^\beta}$. As $\mathrm{d}\,y$ is also eigenvector with eigenvalue α, then for $\alpha \neq 1$ they are linearly independent. For $\alpha = 1$ at least one of $\mathrm{d}\,f$ and $\mathrm{d}\,g$ must be independent with y. Taking this function and y yields a linerizing coordinate change.

Linearization: the last step

This is almost enough for smooth case. For $\alpha = \frac{1}{m}, m \geq 2$ we consider the following Nijenhuis operator.

$$L = \begin{pmatrix} 0 & x + \frac{1}{m} y^m \\ 0 & \frac{1}{m} y \end{pmatrix}.$$

The corresponding vector field is not linearizable, thus the operator is not linearizable.

For $\alpha > 0$ and $\alpha \neq m, \frac{1}{m}, m \geq 2$ the corresponding vector field is always linearizable, thus the Nijenhuis operator L can be brought to linear form.

The only case, that is not covered is $\alpha = 2$. In this case the linearization exists, this requires some work with normal forms of resonant nodes.

Linearization: the last step

The analytic case is more complicated. For $\alpha = -\frac{p}{q}, m, \frac{1}{s}$ for $m \geq 3, s \geq 2$ we have examples of non-linearizable operators (the smooth example works in analytic category as well).

If α is not of this set and also is not in Σ, then the vector fields can be linearized, thus, the analytic linearization exists. The complicated case is $\alpha \in \Sigma$.

It turns out, that in this case there exists a vector field in the form $\xi = (f(x, y), \alpha y)$, such that all its formal linearizations diverge. This is not a simple result, it follows from some deep results on the orbital equivalence and monodromy maps for vector fields on the plane.

Non-degenerate left-symmetric algebras

We have established that tangent space to a singular point of scalar type p possesses a natural structure of left-symmetric algebra. We say that left-symmetric algebra is **non-degenerate** if each Nijenhuis operator with linear part L_1, associated to this algebra, can be linearized. Otherwise the left-symmetric algebra is called degenerate.

Of course, one should distinguish non-degenerate algebras in smooth, analytic and formal category. This definition is due to the similar approach of Weinstein in theory of linearization of Poisson brackets.

Example: In lecture 10 we have shown that left symmetric algebra $\mathfrak{c}_5^+$ with

$$L_1 = \begin{pmatrix} x & y \\ y & x \end{pmatrix}$$

is non-degenerate in both analytic and smooth category.

Non-degenerate left-symmetric algebras

Example: Define the set Σ_{sm} to contain $\alpha < 0$, $\alpha = \frac{1}{m}$ for $m \geq 2$ and s for $s \geq 3$.

Throughout this lecture we have shown that left-symmetric algebra $\mathfrak{b}_{1,\alpha}$ with

$$L_1 = \begin{pmatrix} 0 & x \\ 0 & \alpha y \end{pmatrix}$$

is non-degenerate in smooth case for $\alpha \notin \Sigma_{\mathrm{sm}}$ and degenerate otherwise.

Example: Define the set Σ_{an} to contain $\alpha = -\frac{p}{q}$, Σ (negative irrational numbers that are not Brjuno numbers), $\alpha = \frac{1}{m}$ for $m \geq 2$ and s for $s \geq 3$.

Throughout this lecture we have (almost) shown that left-symmetric algebra $\mathfrak{b}_{1,\alpha}$ with the same L_1 as in previous example is non-degenerate in analytic case for $\alpha \notin \Sigma_{\mathrm{an}}$ and degenerate otherwise.

Non-degenerate left-symmetric algebras

Example: In the end of the Lecture 10 we have shown that left-symmetric algebra $\mathfrak{c}_4$ with

$$L_1 = \begin{pmatrix} y & x \\ 0 & y \end{pmatrix}$$

is degenerate in both smooth and analytic category. We have written the following perturbation

$$\begin{pmatrix} y & x \\ 0 & y \end{pmatrix} + \begin{pmatrix} yx^2 & x^3 \\ -xy^2 & -yx^2 \end{pmatrix}$$

Linearization in dimension two: smooth case

Theorem (Smooth case)

For dimension two in smooth category the following table holds

Degenerate LSA	*Non-degenerate LSA*
$\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{c}_3, \mathfrak{c}_4,$ $\mathfrak{b}_5, \mathfrak{b}_{2,\beta}$ $\mathfrak{b}_{1,\alpha}$ for $\alpha \in \Sigma_{\mathrm{sm}}$	$\mathfrak{b}_4^+, \mathfrak{b}_4^-, \mathfrak{c}_5^+, \mathfrak{c}_5^-$ $\mathfrak{b}_3, \mathfrak{b}_{1,\alpha}$ for $\alpha \notin \Sigma_{\mathrm{sm}}$

Theorem (Analytic case)

For dimension two in analytic category the following table holds

Degenerate LSA	*Non-degenerate LSA*
$\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{c}_3, \mathfrak{c}_4,$ $\mathfrak{b}_5, \mathfrak{b}_{2,\beta}$ $\mathfrak{b}_{1,\alpha}$ for $\alpha \in \Sigma_{\mathrm{an}}$	$\mathfrak{b}_4^+, \mathfrak{b}_4^-, \mathfrak{c}_5^+, \mathfrak{c}_5^-$ $\mathfrak{b}_3, \mathfrak{b}_{1,\alpha}$ for $\alpha \notin \Sigma_{\mathrm{an}}$