Nijenhuis Geometry

Alexey Bolsinov and Andrey Konyaev

Lecture 12:
gl-regular Nijenhuis operators

Sino-Russian Mathematical Center
Moscow University and Peking University

Spring 2021

to contact me: bolsinov@mail.ru

Outline of Lecture 12

- What is a generic/typical operator?
- gl-regular operators and their properties
- Diagonalisation vs companion form
- First and second companion forms for Nijenhuis operators
- Compatibility conditions for PDE systems
- Exercises

What is a generic/typical square matrix?

Remember: matrices may have different meaning

- matrix of a system of linear equations $A\mathbf{x} = \mathbf{b}$ or, equivalently, matrix of a linear operator $A : V \to U$;
- matrix B of a bilinear form (symmetric or skew-symmetric)
- matrix of a linear operator $L : V \to V$

Four natural requirements:

- set of "generic matrices" must be open and dense
- they should share common essential properties
- condition of being "generic" must be invariant
- algebraic and easy to check

Generic matrices?

For a system of linear equations $A\mathbf{x} = \mathbf{b}$, the answer is well known:

$$\det A \neq 0.$$

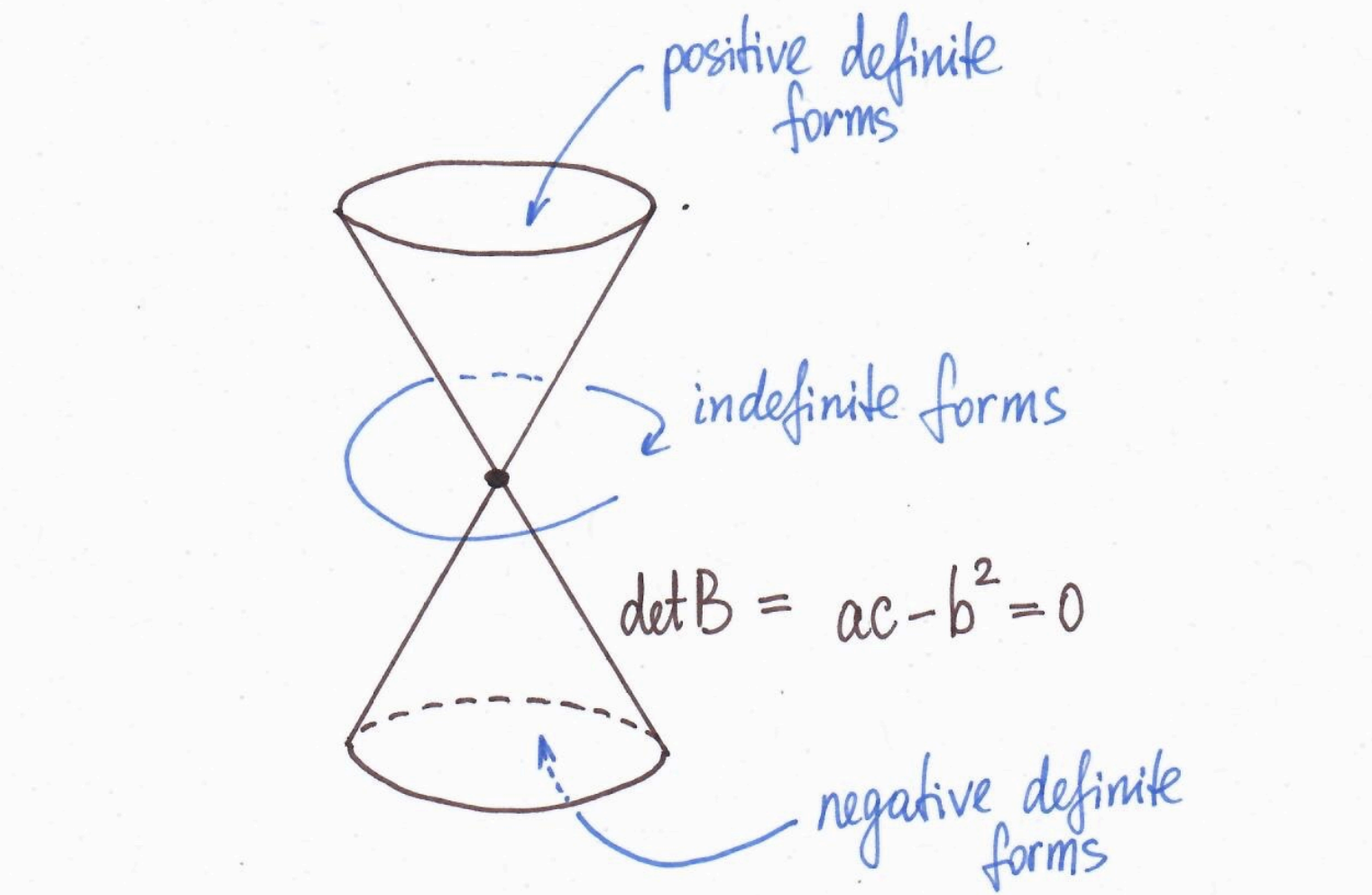
Indeed, all matrices satisfying this condition can be reduced (by using Gaussian elimination) to the identity matrix Id. They form an open dense subset, share many common properties and, what is very important, are very different from degenerate matrices with $\det A = 0$.

For a skew-symmetric bilinear form (in even-dimensional space), the situation is similar. If such a form is non-degenerate, then its matrix reduces to $B = \begin{pmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{pmatrix}$, and therefore, the only reasonable condition is again $\det B \neq 0$.

For a symmetric bilinear form, the situation is slightly different. Still the condition $\det B \neq 0$ is most important, but there are subcases (illustrated on the next slide). Consider the simplest 2×2 case:

$$\left\{ B = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \right\} \simeq \mathbb{R}^3(a, b, c)$$

Types (orbits) of symmetric matrices



Here we can clearly see all the possibilities: (1) positive definite, (2) negative definite, (3) positive semidefinite, (4) negative semidefinite, (5) indefinite, (6) trivial, i.e., zero form.

Generic bilinear forms vs generic operators

Thus, typical/generic bilinear forms (symmetric or skew-symmetric) should satisfy the condition

$$\det B \neq 0.$$

That is exactly what we assume in Riemannian and symplectic geometries:

$$\det(g_{ij}) \neq 0 \qquad \text{and} \qquad \det(\omega_{ij}) \neq 0.$$

Notice that this conditions distinguish one or several open orbits in the space of symmetric or skew-symmetric matrices.

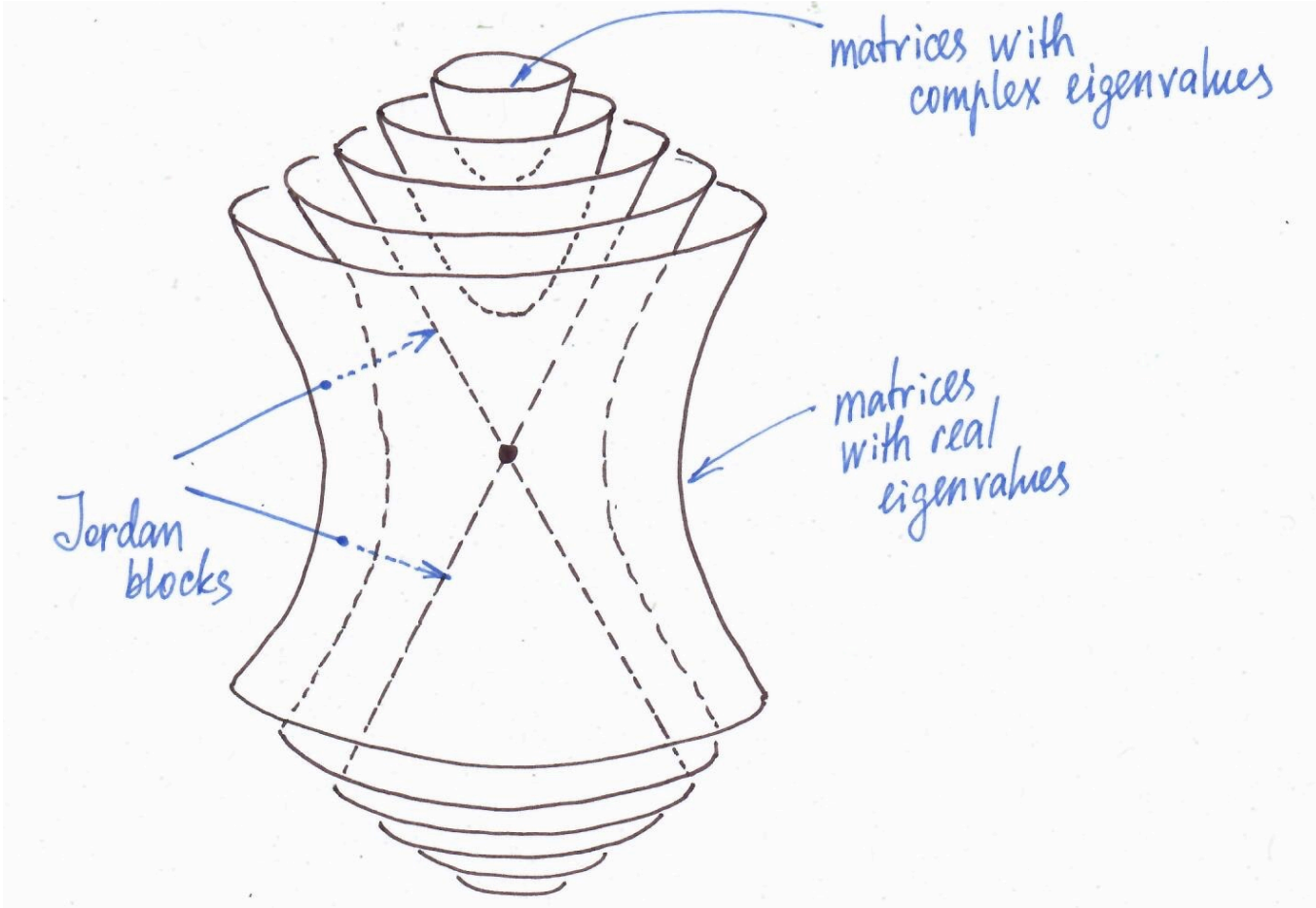
In the case of operators, $\det L \neq 0$ makes sense, but not very natural...

The invariance condition says that "being generic" is related to the natural action of $\mathrm{GL}(n, \mathbb{R})$ (group of coordinate transformations):

$$L \mapsto PLP^{-1}, \qquad P \in \mathrm{GL}(n, \mathbb{R}).$$

In other words, "generic" is a condition not on the operator itself, but on its orbit.

Types (orbits) of traceless matrices/operators



Partition of $sl(2,\mathbb{R})$, space of traceless operators, into orbits is defined by

$$\det \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = -a^2 - bc = \text{const}.$$

"Good" matrices have 2-dimensional orbits. The only "bad" matrix is the zero matrix.

What is a gl-regular operator (matrix)?

Definition

A linear operator $L : \mathbb{R}^n \to \mathbb{R}^n$ is called gl-regular, if one of the following equivalent conditions hold

- the dimension of its orbit $\mathcal{O}(L) = \{ALA^{-1} \mid A \in \mathrm{GL}(n, \mathbb{R})\}$ is maximal (and equals $n^2 - n$)
- $\mathrm{Id}, L, L^2, \ldots, L^{n-1}$ linearly independent
- minimal polynomial of L coincides with characteristic one (or $\deg \chi_{\min} = n$)
- Jordan normal form of L contains exactly one Jordan block for each eigenvalue
- the geometric multiplicity of each eigenvalue is one
- there exist a vector ξ such that $\xi, L\xi, \ldots, L^{n-1}\xi$ form a basis
- the centraliser $\mathcal{C}(L) = \{B \ : \ BL = LB\}$ of L is generated by $\mathrm{Id}, L, L^2, \ldots, L^{n-1}$
- the centraliser $\mathcal{C}(L)$ is commutative

What is a gl-regular operator (continued...)?

- the dimension of the centraliser $\mathcal{C}(L)$ is minimal and equals n
- L can be reduced to the form

$$L_{\text{comp1}} = \begin{pmatrix} \sigma_1 & 1 & \dots & 0 \\ \vdots & & \ddots & \\ \sigma_{n-1} & & & 1 \\ \sigma_n & 0 & \dots & 0 \end{pmatrix}$$

- as well as to

$$L_{\text{comp2}} = \begin{pmatrix} 0 & 1 & \dots & 0 \\ \vdots & & \ddots & \\ 0 & & & 1 \\ \sigma_n & \sigma_{n-1} & \dots & \sigma_1 \end{pmatrix}$$

- the coefficients $\sigma_1, \dots, \sigma_n$ of the characteristic polynomial are functionally independent at $L \in \mathrm{gl}(n)$
- the functions $\tau_1 = \operatorname{tr} L, \tau_2 = \operatorname{tr} L^2, \dots, \tau_n = \operatorname{tr} L^n$ are functionally independent at $L \in \mathrm{gl}(n)$

Important property. The complement to the set of gl-regular operators has codimension 3.

Diagonalisation vs companion form

Given a matrix L of a linear operator, we want to change a basis to reduce L to a simpler (canonical form).
The standard approach is to (try to) diagonalise L by means of a suitable similarity transformation:

$$L \mapsto P^{-1} L\ P = D = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}.$$

If diagonalisation is impossible, then D is "replaced" by Jordan normal form.
For gl-regular matrices, we can use another canonical form

$$L \mapsto \widetilde{P}^{-1} L\ \widetilde{P} = L_{\text{comp1}} = \begin{pmatrix} \sigma_1 & 1 & \dots & 0 \\ \vdots & & \ddots & \\ \sigma_{n-1} & & & 1 \\ \sigma_n & 0 & \dots & 0 \end{pmatrix}.$$

Let us compare these two forms D and L_{comp1}. Which one is "better"?

Diagonalisation vs companion form

- Both D and L_{comp1} depend on n parameters

$$\lambda_1, \ldots, \lambda_n \quad \text{and} \quad \sigma_1, \ldots, \sigma_n$$

- These parameters are related in a very natural way: λ_i are the roots of the polynomial $t^n - \sigma_1 t^{n-1} - \cdots - \sigma_{n-1} t - \sigma_n$.
- For D, there are some issues with complex eigenvalues. No such problems with L_{comp1}, which is a real matrix.
- To find λ_i we first need to compute the characteristic polynomial of L, then find its roots (which is a very difficult task...). No such problem for L_{comp1}.
- Diagonalisation: to find the transition matrix P we need to find an eigenvector e_i for each eigenvalue λ_i by solving $(L - \lambda_i \cdot \mathrm{Id})\mathbf{x} = \mathbf{0}$.
- Companion form: to find the transition matrix $\widetilde{P}$ or, equivalently, the corresponding new basis vectors, we simply take $\tilde{e}_n = \xi$, $\tilde{e}_{n-1} = L\tilde{e}_n$, $\ldots$, $\tilde{e}_1 = L^{n-1}\tilde{e}_1$ with ξ being an (arbitrary) generic vector.
- Canonical basis in both cases is not unique, but depends on n parameters (rescaling of each eigenvector e_i and choosing ξ)
- Diagonal form D is much better for computations like D^n, $\exp D$, etc.

What do we want to know about gl-regular Nijenhuis operators?

Definition

A Nijenhuis operator $L(x)$ (on a manifold M) is called gl-*regular*, if it is gl-regular at each point $x \in M$.

Natural questions:

- Local canonical forms and local classification
- Perturbation of a Jordan block (in the class of Nijenhuis operators): which scenarios are allowed?
- Structure of the set of singular points
- Topological obstructions for existence of gl-regular Nijenhuis operators on compact manifolds
- Examples of regular Nijenhuis operators on compact manifolds

First companion form

Theorem

Let L be a real analytic gl-regular Nijenhuis operator with characteristic polynomial $\chi_L(\lambda) = \det(\lambda \cdot \mathrm{Id} - L) = \lambda^n - f_1\lambda^{n-1} - \dots - f_n$. Then in a suitable coordinate system $x = (x^1, \dots, x^n)$, this operator takes the form

$$L_{\mathrm{comp1}}(x) = \begin{pmatrix} f_1 & 1 & 0 & \dots & 0 \\ f_2 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ f_{n-1} & 0 & \dots & 0 & 1 \\ f_n & 0 & \dots & 0 & 0 \end{pmatrix}, \tag{1}$$

where $f_i = f_i(x)$ are the coefficients of the characteristic polynomial in this coordinate system. They satisfy the following PDE system:

$$\begin{aligned} \frac{\partial f_i}{\partial x^j} &= f_i \frac{\partial f_1}{\partial x^{j+1}} + \frac{\partial f_{i+1}}{\partial x^{j+1}}, \\ \frac{\partial f_n}{\partial x^j} &= f_n \frac{\partial f_1}{\partial x^{j+1}}. \end{aligned} \qquad i, j = 1, \dots, n-1. \tag{2}$$

Second companion form

Theorem

Let L be a real analytic gl-regular Nijenhuis operator with characteristic polynomial $\chi_L(\lambda) = \det(\lambda \cdot \mathrm{Id} - L) = \lambda^n - f_1\lambda^{n-1} - \cdots - f_n$. Then in a suitable coordinate system $x = (x^1, \ldots, x^n)$, this operator takes the form

$$L_{\mathrm{comp2}}(x) = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \ldots & 0 & 1 \\ f_n & f_{n-1} & \ldots & f_2 & f_1 \end{pmatrix}, \tag{3}$$

where $f_i = f_i(x)$ are the coefficients of the characteristic polynomial in this coordinate system. They satisfy the PDE system that can be written in the form

$$\mathrm{d}\,\omega = 0, \quad \mathrm{d}\left(L^*\omega\right) = 0, \tag{4}$$

where $\omega = f_n \mathrm{d}\,x^1 + \cdots + f_1 \mathrm{d}\,x^n$.

Reducibility to a companion form is a strong condition

Question. Why not every operator gl-regular operator can be reduced to a companion form? In other words, does the Nijenhuis condition is really important?

Explanation. Companion form is parametrised by n functions (in n variables each). A coordinate change $y \mapsto x = x(y)$ is also parametrised by n functions (in n variables). However, an operator field L (which is not necessarily Nijenhuis) is parametrised by n^2 functions (in n variables).

Thus, a coordinate transformation can "eliminate" n functions, but $n^2 - n$ still survive. Hence, if $n^2 - n > n$ (i.e. starting from dimension 3) we can always find an operator that cannot be reduced to a companion form.

Example. The operator $L = \begin{pmatrix} 0 & 1 & 0 \\ -(y^2+1) & 0 & 1 \\ 0 & y^2+1 & 0 \end{pmatrix}$ defined in $\mathbb{R}^3(x, y, z)$ cannot be reduced to a companion form. Indeed, L is nilpotent (verify this!) and its companion form is the standard Jordan 0-block, i.e., a constant matrix. Hence, reducibility to companion form would imply that L is Nijenhuis, but it is not (verify this!).

Comments

- "Canonical" coordinate system $x = (x^1, \dots, x^n)$ is not unique. Moreover, the functions $f_1, \dots, f_n$ in the first column (last row) are not unique either.
- Formulas (1) and (3) should be viewed as pre-canonical forms. Local classification problem is open.
- Formulas (1) and (3) work both at generic and singular points. In particular $f_1, \dots, f_n$ may have a very complicated singularity.
- If L is differentially non-degenerate Nijenhuis operator (i.e., $f_1, \dots, f_n$ are independent), then there exist "distinguished" coordinate systems:
  - in the first formula $x^1 = f_1\ ,\ x^2 = f_2\ ,\ \dots\ ,\ x^n = f_n$,
  - in the second formula $x^1 = \operatorname{tr} L, x^2 = \frac{1}{2}\operatorname{tr} L^2, \dots, x^n = \frac{1}{n}\operatorname{tr} L^n$.
- (1) is equivalent to the existence of a vector field ξ such that $\xi, L\xi, \dots, L^{n-1}\xi$ are linearly independent and commute.
- (3) is equivalent to the existence of a function h such that the forms $\mathrm{d}h, L^*\mathrm{d}h, \dots, (L^*)^{n-1}\mathrm{d}h$ are linearly independent and closed.

Proof of the first companion form theorem

We first check that

$$L_{\text{comp1}}(x) = \begin{pmatrix} f_1 & 1 & 0 & \dots & 0 \\ f_2 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ f_{n-1} & 0 & \dots & 0 & 1 \\ f_n & 0 & \dots & 0 & 0 \end{pmatrix},$$

is Nijenhuis if and only if the relations

$$\frac{\partial f_i}{\partial x^j} = f_i \frac{\partial f_1}{\partial x^{j+1}} + \frac{\partial f_{i+1}}{\partial x^{j+1}},$$
$$\frac{\partial f_n}{\partial x^j} = f_n \frac{\partial f_1}{\partial x^{j+1}}.$$

hold.

We need to check the Nijenhuis relation

$$L^2[e_i, e_j] - L[Le_i, e_j] - L[e_i, Le_j] + [Le_i, Le_j] = 0$$

for any pair of basis vectors $e_i = \partial_{x^i}$, $e_j = \partial_{x^j}$ and $L = L_{\text{comp1}}$.
Obvious for $i, j > 1$.

We first notice that our system of PDEs can be rewritten in the following matrix form (to simplify the notation, we denote $L = L_{\text{comp1}}$)

$$\begin{pmatrix} \frac{\partial f_1}{\partial x^j} \\ \vdots \\ \frac{\partial f_n}{\partial x^j} \end{pmatrix} = L \begin{pmatrix} \frac{\partial f_1}{\partial x^{j+1}} \\ \vdots \\ \frac{\partial f_n}{\partial x^{j+1}} \end{pmatrix} \tag{5}$$

Next, we take $e_1 = \partial_{x^1}$ and $e_{j+1} = \partial_{x^{j+1}}$ to verify the Nijenhuis relation:

$$0 = L^2[\partial_{x^1}, \partial_{x^{j+1}}] - L[\partial_{x^1}, L\partial_{x^{j+1}}] - L[L\partial_{x^1}, \partial_{x^{j+1}}] + [L\partial_{x^1}, L\partial_{x^{j+1}}] =$$

(we use $L\partial_{x^{j+1}} = \partial_{x^j}$, $L\partial_{x^1} = \sum f_i \partial_{x^i}$)

$$-L[\partial_{x^1}, \partial_{x^j}] - L\left[\sum f_i \partial_{x^i}, \partial_{x^{j+1}}\right] + \left[\sum f_i \partial_{x^i}, \partial_{x^j}\right] =$$

$$L\left(\sum \frac{\partial f_i}{\partial_{x^{j+1}}} \partial_{x^i}\right) - \sum \frac{\partial f_i}{\partial_{x^j}} \partial_{x^i} = L\left(\sum \frac{\partial f_i}{\partial_{x^{j+1}}} e_i\right) - \sum \frac{\partial f_i}{\partial_{x^j}} e_i$$

which is equivalent, in matrix form, to $L \begin{pmatrix} \frac{\partial f_1}{\partial x^{j+1}} \\ \vdots \\ \frac{\partial f_n}{\partial x^{j+1}} \end{pmatrix} - \begin{pmatrix} \frac{\partial f_1}{\partial x^j} \\ \vdots \\ \frac{\partial f_n}{\partial x^j} \end{pmatrix} = 0$, as needed.

Proof continued...

Consider a gl-regular Nijenhuis operator L given is local coordinates $u = (u^1, \dots, u^n)$. Our goal is to find coordinate transformation $u = u(x)$, $x = (x^1, \dots, x^n)$, bringing L to the first companion form (1).
This leads us to the following system of PDEs:

$$\left(\frac{\partial u}{\partial x}\right)^{-1} L(u) \left(\frac{\partial u}{\partial x}\right) = L_{\text{comp1}}(x), \tag{6}$$

where L_{comp1} stands for the first companion form (1) and $\left(\frac{\partial u}{\partial x}\right)$ denotes the Jacobi matrix of the transformation $u = u(x)$:

$$\left(\frac{\partial u}{\partial x}\right) = \begin{pmatrix} u^1_{x^1} & \dots & u^1_{x^n} \\ \vdots & \ddots & \vdots \\ u^n_{x^1} & \dots & u^n_{x^n} \end{pmatrix}.$$

Since our transformation is invertible, we rewrite (6) as

$$\left(\frac{\partial u}{\partial x}\right) L_{\text{comp1}} = L \left(\frac{\partial u}{\partial x}\right) \tag{7}$$

That is the system of (quasilinear) PDEs we need to solve.

Here and below, when doing matrix computation, we consider u and x as column-vectors, also we use u_{x^i} or $u^j_{x^i}$ for partial derivatives.

Lemma

System (7), *i.e.,* $\left(\frac{\partial u}{\partial x}\right) L_{\text{comp1}} = L\left(\frac{\partial u}{\partial x}\right)$ *is equivalent to*

$$u_{x^k} = L u_{x^{k+1}}, \ \textit{or, equivalently } u_{x^{n-k}} = L^k u_{x^n}, \ k = 1, \dots, n-1. \tag{8}$$

Proof. By construction, (8) simply means that all the columns of the matrices in the left and right hand sides of (7) coincides except for the first column. In other words, system (7) contains one additional vector relation for the first columns of l.h.s. and r.h.s. of (7), namely:

$$f_1 u_{x^1} + f_2 u_{x^2} + \cdots + f_n u_{x^n} = L u_{x^1}. \tag{9}$$

We need to show that this relation follows from (8). Substituting $u_{x^{n-k}} = L^k u_{x^n}$ into (9) gives

$$f_1 L^{n-1} u_{x^n} + f_2 L^{n-2} u_{x^n} + \cdots + f_n u_{x^n} = L^n u_{x^n}$$

or equivalently

$$\left(L^n - f_1 L^{n-1} - f_2 L^{n-2} - \cdots - f_n \operatorname{Id}\right) u_{x^n} = \chi_L(L) u_{x^n} = 0,$$

which holds true automatically by the Cayley–Hamilton theorem.

Additional comments for the proof of Lemma

$$\left(\frac{\partial u}{\partial x}\right) \llcorner_{comp} = \llcorner \left(\frac{\partial u}{\partial x}\right) \qquad (7)$$

$$\begin{pmatrix} \frac{\partial u^1}{\partial x^1} & \cdots & \frac{\partial u^1}{\partial x^n} \\ \vdots & & \vdots \\ \frac{\partial u^n}{\partial x^1} & & \frac{\partial u^n}{\partial x^n} \end{pmatrix} \begin{pmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix} = \llcorner \begin{pmatrix} \frac{\partial u^1}{\partial x^1} & \cdots & \frac{\partial u^1}{\partial x^n} \\ \vdots & & \vdots \\ \frac{\partial u^n}{\partial x^1} & \cdots & \frac{\partial u^n}{\partial x^n} \end{pmatrix}$$

consider the last column

$$\begin{pmatrix} \frac{\partial u^1}{\partial x^{n-1}} \\ \vdots \\ \frac{\partial u^n}{\partial x^{n-1}} \end{pmatrix} = \llcorner \begin{pmatrix} \frac{\partial u^1}{\partial x^n} \\ \vdots \\ \frac{\partial u^n}{\partial x^n} \end{pmatrix}, \text{ i.e. } u_{x^{n-1}} = \llcorner u_{x^n}$$

and so on

Proof continued...

Thus, we are led to the following system of (quasilinear) PDEs

$$u_{x^{n-k}} = L^k u_{x^n}, \;\; k = 1, \ldots, n-1.$$

where $L^k = L \cdot L \ldots L$ (k times) and L is a Nijenhuis operator.

This system is overdetermined and (for arbitrary L) is not necessarily consistent.

What can we say about this kind of PDE systems?

The key property

Consider the system of quasilinear PDEs (of hydrodynamic type):

$$u_{x^k} = A_k(u)u_{x^n}, \quad k = 1, \dots, n-1. \tag{10}$$

Proposition

The following properties of (10) are equivalent

(A) For any real analytic initial condition

$$\begin{aligned} u^1(0, \dots, 0, x^n) &= h^1(x^n), \\ u^2(0, \dots, 0, x^n) &= h^2(x^n), \\ &\dots \\ u^n(0, \dots, 0, x^n) &= h^n(x^n), \end{aligned} \qquad \text{or shortly} \quad u(0, \dots, 0, x^n) = h(x^n),$$

where h is a real analytic vector-function of one variable, there exists a unique real analytic solution $u = u(x)$ of (10).

(B) Operators A_k's pairwise commute (i.e., $A_kA_j = A_jA_k$) and

$$\langle A_k, A_j \rangle(\xi, \xi) \stackrel{\text{def}}{=} [A_k\xi, A_j\xi] - A_j[A_k\xi, \xi] - A_k[\xi, A_j\xi] = 0 \tag{11}$$

for any vector field ξ and $k, j = 1, \dots, n-1$.

Exercises

Exercise

Prove that the properties from the definition of a gl-regular operator are equivalent.

Exercise

Let $L(x)$ be an operator field (not necessarily Nijenhuis) and $\sigma_1, \ldots, \sigma_n$ be the coefficients of its characteristic polynomial. Prove that if L is not gl-regular at a point p, then the differentials $\mathrm{d}\,\sigma_1(p), \ldots, \mathrm{d}\,\sigma_n(p)$ are linearly dependent.

Exercise

Prove that the existence of the first companion form (1) is equivalent to the existence of a vector field ξ such that $\xi, L\xi, \ldots, L^{n-1}\xi$ are linearly independent and commute.

Exercise

Prove the the existence of the second companion form (3) is equivalent to the existence of a function h such that the forms $\mathrm{d}h, L^*\mathrm{d}h, \ldots, (L^*)^{n-1}\mathrm{d}h$ are linearly independent and closed.