Nijenhuis Geometry

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Lecture 13:
gl-regular Nijenhuis operators (part 2)

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Outline of Lecture 12

- gl-regularity (reminder)
- Proof of "Companion Form Theorem"
- Compatibility conditions for PDEs
- Local description of gl-regular Nijenhuis operators (not local classification yet)
- Algebraic formulas for solving PDEs (of very special kind).
- Exercises

What is a gl-regular operator (matrix)?

Four natural requirements:

- set of "generic operators" must be open and dense
- they should share common essential properties
- condition of being "generic" must be invariant
- algebraic and easy to check

Definition 1

A linear operator $L : \mathbb{R}^n \to \mathbb{R}^n$ is called gl-regular, if one of the following equivalent conditions hold

- the dimension of its orbit $\mathcal{O}(L) = \{ALA^{-1} \mid A \in \mathrm{GL}(n, \mathbb{R})\}$ is maximal (and equals $n^2 - n$)
- $\mathrm{Id}, L, L^2, \ldots, L^{n-1}$ linearly independent
- Jordan normal form of L contains exactly one Jordan block for each eigenvalue
- there exist a vector ξ such that $\xi, L\xi, \ldots, L^{n-1}\xi$ form a basis

What is a gl-regular operator (continued...)?

- L can be reduced to the form

$$L_{\text{comp1}} = \begin{pmatrix} \sigma_1 & 1 & \dots & 0 \\ \vdots & & \ddots & \\ \sigma_{n-1} & & & 1 \\ \sigma_n & 0 & \dots & 0 \end{pmatrix}$$

- as well as to

$$L_{\text{comp2}} = \begin{pmatrix} 0 & 1 & \dots & 0 \\ \vdots & & \ddots & \\ 0 & & & 1 \\ \sigma_n & \sigma_{n-1} & \dots & \sigma_1 \end{pmatrix}$$

- the coefficients $\sigma_1, \dots, \sigma_n$ of the characteristic polynomial are functionally independent at $L \in \mathrm{gl}(n)$
- the functions $\tau_1 = \operatorname{tr} L, \tau_2 = \operatorname{tr} L^2, \dots, \tau_n = \operatorname{tr} L^n$ are functionally independent at $L \in \mathrm{gl}(n)$

Important property. The complement to the set of gl-regular operators has codimension 3.

gl-regular Nijenhuis operators

Definition 2

A Nijenhuis operator $L(x)$ (on a manifold M) is called gl-*regular*, if it is gl-regular at each point $x \in M$.

Natural questions:

- Local canonical forms and local classification
- Perturbation of a Jordan block (in the class of Nijenhuis operators): which scenarios are allowed?
- Structure of the set of singular points
- Topological obstructions for existence of gl-regular Nijenhuis operators on compact manifolds
- Examples of regular Nijenhuis operators on compact manifolds

First companion form

Theorem 3

Let L be a real analytic gl-regular Nijenhuis operator with characteristic polynomial $\chi_L(\lambda) = \det(\lambda \cdot \mathrm{Id} - L) = \lambda^n - f_1\lambda^{n-1} - \dots - f_n$. Then in a suitable coordinate system $x = (x^1, \dots, x^n)$, this operator takes the form

$$L_{\mathrm{comp1}}(x) = \begin{pmatrix} f_1 & 1 & 0 & \dots & 0 \\ f_2 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ f_{n-1} & 0 & \dots & 0 & 1 \\ f_n & 0 & \dots & 0 & 0 \end{pmatrix}, \tag{1}$$

where $f_i = f_i(x)$ are the coefficients of the characteristic polynomial in this coordinate system. They satisfy the following PDE system:

$$\begin{aligned} \frac{\partial f_i}{\partial x^j} &= f_i \frac{\partial f_1}{\partial x^{j+1}} + \frac{\partial f_{i+1}}{\partial x^{j+1}}, \\ \frac{\partial f_n}{\partial x^j} &= f_n \frac{\partial f_1}{\partial x^{j+1}}. \end{aligned} \qquad i, j = 1, \dots, n-1. \tag{2}$$

Second companion form

Theorem 4

Let L be a real analytic gl-regular Nijenhuis operator with characteristic polynomial $\chi_L(\lambda) = \det(\lambda \cdot \mathrm{Id} - L) = \lambda^n - f_1\lambda^{n-1} - \cdots - f_n$. Then in a suitable coordinate system $x = (x^1, \ldots, x^n)$, this operator takes the form

$$L_{\mathrm{comp2}}(x) = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \ldots & 0 & 1 \\ f_n & f_{n-1} & \ldots & f_2 & f_1 \end{pmatrix}, \tag{3}$$

where $f_i = f_i(x)$ are the coefficients of the characteristic polynomial in this coordinate system. They satisfy the PDE system that can be written in the form

$$\mathrm{d}\,\omega = 0, \quad \mathrm{d}\left(L^*\omega\right) = 0, \tag{4}$$

where $\omega = f_n \mathrm{d}\,x^1 + \cdots + f_1 \mathrm{d}\,x^n$.

Proof of the first companion form theorem

We have already checked the following important fact:

$$L_{\mathrm{comp1}}(x) = \begin{pmatrix} f_1 & 1 & 0 & \dots & 0 \\ f_2 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ f_{n-1} & 0 & \dots & 0 & 1 \\ f_n & 0 & \dots & 0 & 0 \end{pmatrix},$$

is Nijenhuis if and only if the relations

$$\frac{\partial f_i}{\partial x^j} = f_i \frac{\partial f_1}{\partial x^{j+1}} + \frac{\partial f_{i+1}}{\partial x^{j+1}},$$
$$\frac{\partial f_n}{\partial x^j} = f_n \frac{\partial f_1}{\partial x^{j+1}}.$$

hold.

Proof continued...

Consider a gl-regular Nijenhuis operator L given is local coordinates $u = (u^1, \dots, u^n)$.

Our main goal is to find coordinate transformation $u = u(x)$, $x = (x^1, \dots, x^n)$, bringing L to the first companion form (1), i.e., satisfying

$$\boxed{\left(\frac{\partial u}{\partial x}\right)^{-1} L(u) \left(\frac{\partial u}{\partial x}\right) = L_{\mathrm{comp1}}(x),} \tag{5}$$

We have shown that this system of PDEs can be rewritten in the form $u_{x^k} = L u_{x^{k+1}}$, or, equivalently

$$\boxed{u_{x^{n-k}} = L^k u_{x^n}, \quad k = 1, \dots, n-1.} \tag{6}$$

This system is overdetermined and (for arbitrary L) is not necessarily consistent.

What can we say about this kind of PDE systems?

The key property

Consider the system of quasilinear PDEs (of hydrodynamic type):

$$u_{x^k} = A_k(u)u_{x^n}, \quad k = 1, \dots, n-1. \tag{7}$$

Proposition 5

The following properties of (7) are equivalent

(A) For any real analytic initial condition

$$\begin{aligned} u^1(0, \dots, 0, x^n) &= h^1(x^n), \\ u^2(0, \dots, 0, x^n) &= h^2(x^n), \\ &\dots \\ u^n(0, \dots, 0, x^n) &= h^n(x^n), \end{aligned} \qquad \text{or shortly} \quad u(0, \dots, 0, x^n) = h(x^n),$$

where h is a real analytic vector-function of one variable, there exists a unique real analytic solution $u = u(x)$ of (7).

(B) Operators A_k's pairwise commute (i.e., $A_kA_j = A_jA_k$) and

$$\langle A_k, A_j \rangle(\xi, \xi) \stackrel{\text{def}}{=} [A_k\xi, A_j\xi] - A_j[A_k\xi, \xi] - A_k[\xi, A_j\xi] = 0 \tag{8}$$

for any vector field ξ and $k, j = 1, \dots, n-1$.

What is the meaning of Condition (B)?

Condition (B) is known as compatibility condition for (7). To explain the meaning of (B), consider two equations from our system:

$u_{x^k} = A_k(u)u_{x^n}$ and $u_{x^j} = A_j(u)u_{x^n}$ or, simpler, $u_x = Au_z$ and $u_y = Bu_z$

and write down the relation $\partial_y u_x = \partial_x u_y$ for the r.h.s of these equations:

$$\partial_y\left(Au_z\right) = \partial_y\left(A^s_\alpha u^\alpha_z\right) =$$

$$\frac{\partial A^s_\alpha}{\partial u^\beta} u^\beta_y u^\alpha_z + A^s_\alpha u^\alpha_{yz} = \frac{\partial A^s_\alpha}{\partial u^\beta} B^\beta_\gamma u^\gamma_z u^\alpha_z + A^s_\alpha \partial_z(B^\alpha_\gamma u^\gamma_z) =$$

$$\frac{\partial A^s_\alpha}{\partial u^\beta} B^\beta_\gamma u^\gamma_z u^\alpha_z + A^s_\alpha \frac{\partial B^\alpha_\gamma}{\partial u^\beta} u^\beta_z u^\gamma_z + A^s_\alpha B^\alpha_\gamma u^\gamma_{zz} =$$

$$\left(\frac{\partial A^s_\alpha}{\partial u^\beta} B^\beta_\gamma + A^s_\beta \frac{\partial B^\beta_\gamma}{\partial u^\alpha}\right) u^\gamma_z u^\alpha_z + A^s_\alpha B^\alpha_\gamma u^\gamma_{zz}$$

Hence, $\partial_y u_x = \partial_x u_y$ means (we interchange A and B and subtract)

$$\left(\frac{\partial A^s_\alpha}{\partial u^\beta} B^\beta_\gamma + A^s_\beta \frac{\partial B^\beta_\gamma}{\partial u^\alpha} - \frac{\partial B^s_\alpha}{\partial u^\beta} A^\beta_\gamma + B^s_\beta \frac{\partial A^\beta_\gamma}{\partial u^\alpha}\right) u^\gamma_z u^\alpha_z - (A^s_\alpha B^\alpha_\gamma - B^s_\alpha A^\alpha_\gamma) u^\gamma_{zz} \equiv 0$$

Condition (B) amounts to vanishing of the two expressions in brackets.

Proof of Proposition 5

We will prove a slightly simpler (low-dimensional) version of it. Consider

$$f_x = h_1(x, y, z, f, f_z) \quad \text{and} \quad f_y = h_2(x, y, z, f, f_z) \tag{9}$$

Proposition 6

The following properties of (9) are equivalent

(A) For any real analytic initial condition

$$f(a_1, a_2, z) = v(z)$$

where v is a real analytic, there exists a unique real analytic solution $f = f(x, y, z)$ of (9).

(B) The condition $D_x h_2 = D_y h_1$ holds identically in virtue of the system.

Proof. (A) $\Rightarrow$ (B) is easy.
Assume that (B) holds. In more detail,

$$\frac{\partial h_2}{\partial x} + \frac{\partial h_2}{\partial f} h_1 + \frac{\partial h_2}{\partial f_z}\left(\frac{\partial h_1}{\partial z} + \frac{\partial h_1}{\partial f} f_z\right) = \frac{\partial h_1}{\partial y} + \frac{\partial h_1}{\partial f} h_2 + \frac{\partial h_1}{\partial f_z}\left(\frac{\partial h_2}{\partial z} + \frac{\partial h_2}{\partial f} f_z\right)$$

Without loss of generality, we assume $a_1 = a_2 = 0$ in the initial condition. For given initial condition $f(0, 0, z) = v(z)$ we construct the desired solution in two steps. We first construct a unique function of two variables $u(y, z)$ that is a local real analytic solution of the Cauchy problem

$$u_y = h_2(0, y, z, u, u_z), \quad u(0, z) = v(z).$$

The existence of $u(y, z)$ follows from the classical Cauchy-Kovalevskaya theorem. If the desired solution f of (9) exists then necessarily $f(0, y, z) = u(y, z)$.
The next step is solving the Cauchy problem

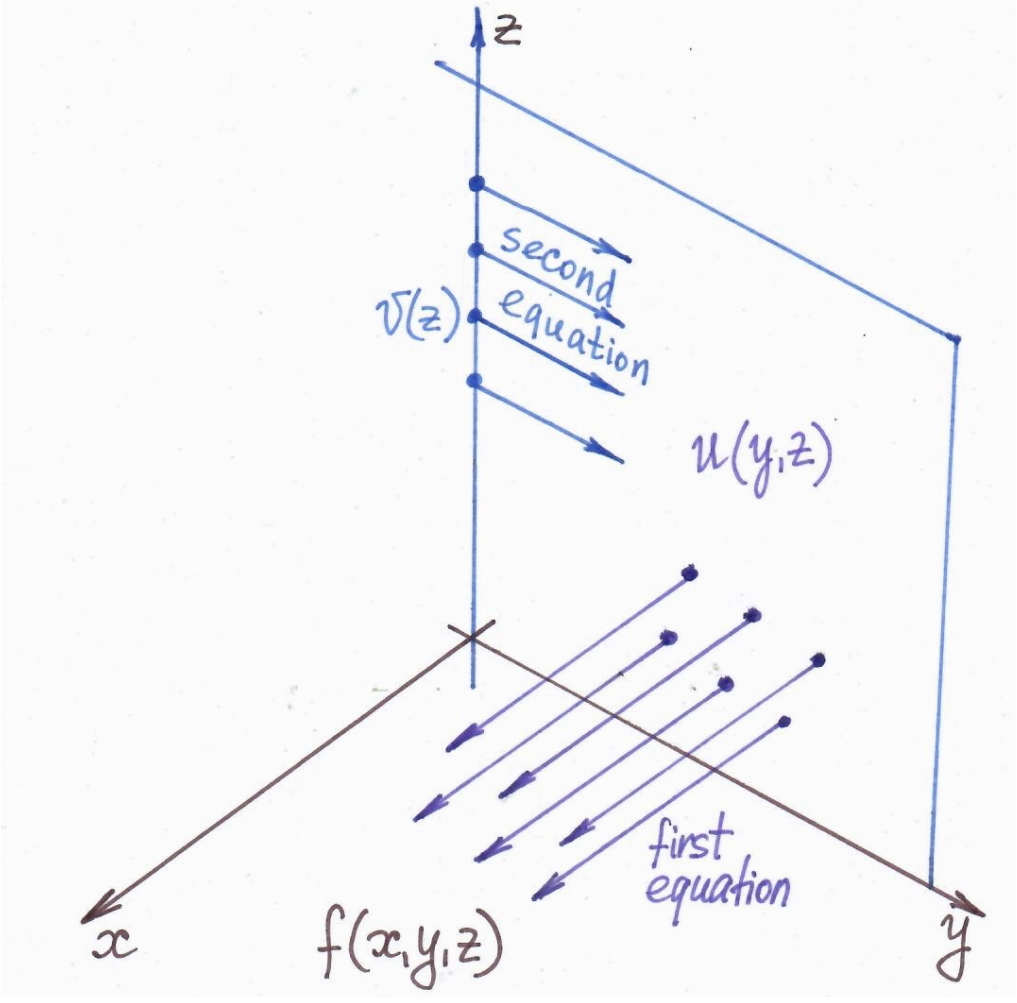
$$f_x = h_1(x, y, z, f, f_z), \quad f(0, y, z) = u(y, z).$$

Again, a unique local real analytic solution $f(x, y, z)$ exists by Cauchy-Kovalevskaya theorem.

Thus, we have constructed a function that satisfies the initial condition and solves the first equation of (9). However the second equation is fulfilled only on the plane $x = 0$.

Our goal is to show that $D_x h_2 = D_y h_1$ guarantees that the second equation holds at all the points.

Solving system in two steps



Lemma 7

Let f satisfy the conditions

$$f_x(x,y,z) = h_1(x,y,z,f,f_z) \quad \textit{and} \quad f_y(0,y,z) = h_2(0,y,z,f,f_z).$$

If the compatibility condition $D_x h_2 = D_y h_1$ is identically fulfilled, then $f_y(x,y,z) = h_2(x,y,z,f,f_z)$, i.e., f is a solution of (9).

Proof. Consider the functions

$$\widehat{h}_2(x,y,z) = h_2(x,y,z,f(x,y,z),f_z(x,y,z)),$$
$$\widehat{h}_1(x,y,z) = h_1(x,y,z,f(x,y,z),f_z(x,y,z)),$$

and compute the partial derivative of $f_y - \widehat{h}_2$ w.r.t. x:

$$\frac{\partial}{\partial x}\left(f_y - \widehat{h}_2\right) = \frac{\partial}{\partial y} f_x - \frac{\partial h_2}{\partial x} - \frac{\partial h_2}{\partial f} f_x - \frac{\partial h_2}{\partial f_z} f_{zx} =$$

$$\frac{\partial \widehat{h}_1}{\partial y} - \frac{\partial h_2}{\partial x} - \frac{\partial h_2}{\partial f}\,\widehat{h}_1 - \frac{\partial h_2}{\partial f_z}\frac{\partial \widehat{h}_1}{\partial z} =$$

$$\frac{\partial h_1}{\partial y} + \frac{\partial h_1}{\partial f} f_y + \frac{\partial h_1}{\partial f_z}(f_y)_z - \frac{\partial h_2}{\partial x} - \frac{\partial h_2}{\partial f}\,\widehat{h}_1 - \frac{\partial h_2}{\partial f_z}\frac{\partial \widehat{h}_1}{\partial z} =$$

and taking into account the compatibility condition:

$$\frac{\partial h_1}{\partial f} f_y + \frac{\partial h_1}{\partial f_z}(f_y)_z - \frac{\partial h_1}{\partial f}\,\widehat{h}_2 - \frac{\partial h_1}{\partial f_z}\frac{\partial \widehat{h}_2}{\partial z} = \frac{\partial h_1}{\partial f}\left(f_y - \widehat{h}_2\right) + \frac{\partial h_1}{\partial f_z}\left(f_y - \widehat{h}_2\right)_z.$$

We see that the function $f_y - \widehat{h}_2$ satisfies the linear PDE:

$$\frac{\partial}{\partial x}\left(f_y - \widehat{h}_2\right) = \frac{\partial h_1}{\partial f}\left(f_y - \widehat{h}_2\right) + \frac{\partial h_1}{\partial f_z}\left(f_y - \widehat{h}_2\right)_z.$$

Moreover for $x = 0$ we have $f_y(0, y, z) - \widehat{h}_2(0, y, z) = 0$. This implies that $f_y - \widehat{h}_2$ vanishes identically, that is,

$$f_y(x, y, z) = h_2(x, y, z, f(x, y, z), f_z(x, y, z)),$$

as stated.

This completes the proof of Proposition 5.

Coming back to our problem:

$$\boxed{u_{x^{n-k}} = L^k u_{x^n}, \quad k = 1, \ldots, n-1,}$$

we need to verify the compatibility condition (B), that is,

- the matrices of these systems commute, i.e., $L^k L^j = L^j L^k$ (which is obvious);
- $\langle L^k, L^j \rangle(\xi, \xi) \stackrel{\text{def}}{=} [L^k\xi, L^j\xi] - L^j[L^k\xi, \xi] - L^k[\xi, L^j\xi] = 0.$

Verifying $\langle L^k, L^j \rangle = 0$ for Nijenhuis operator L

Assume that $j = k + s > k$ and rewrite $[L^k\xi, L^j\xi] - L^j[L^k\xi, \xi] - L^k[\xi, L^j\xi]$ as

$$\mathcal{L}_{L^k\xi}(L^j\xi) + L^j\mathcal{L}_\xi(L^k\xi) - L^k\mathcal{L}_\xi(L^j\xi) =$$

(we use $\mathcal{L}_\eta(A\eta) = (\mathcal{L}_\eta A)\eta + A\,\mathcal{L}_\eta\eta = (\mathcal{L}_\eta A)\eta$)

$$(\mathcal{L}_{L^k\xi}L^s)L^k\xi + L^j(\mathcal{L}_\xi L^k)\xi - L^k(\mathcal{L}_\xi L^j)\xi =$$

$$\Big((\mathcal{L}_{L^k\xi}L^s)L^k + L^j\mathcal{L}_\xi L^k - L^k\mathcal{L}_\xi L^j\Big)\xi =$$

(we use $\mathcal{L}_\xi L^j = \mathcal{L}_\xi(L^sL^k) = L^s\mathcal{L}_\xi L^k + (\mathcal{L}_\xi L^s)L^k$)

$$\Big((\mathcal{L}_{L^k\xi}L^s)L^k + L^{k+s}\mathcal{L}_\xi L^k - L^kL^s\mathcal{L}_\xi L^k - L^k(\mathcal{L}_\xi L^s)L^k\Big)\xi =$$

$$\Big((\mathcal{L}_{L^k\xi}L^s)L^k - L^k(\mathcal{L}_\xi L^s)L^k\Big)\xi = \Big((\mathcal{L}_{L^k\xi} - L^k\mathcal{L}_\xi)L^s\Big)L^k\xi$$

We know that $(\mathcal{L}_{L^k\xi} - L^k\mathcal{L}_\xi)L = 0$ implying $(\mathcal{L}_{L^k\xi} - L^k\mathcal{L}_\xi)L^s = 0$ and hence completing the proof of "Companion Form Theorem".

What next?

Can this "companion form" theorem be understood as local description of gl-regular Nijenhuis operators?
No... because we have no explicit form for the functions $f_1, \dots, f_n$ in the first column of L_{comp1}:

$$L_{\mathrm{comp1}}(x) = \begin{pmatrix} f_1 & 1 & 0 & \dots & 0 \\ f_2 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ f_{n-1} & 0 & \dots & 0 & 1 \\ f_n & 0 & \dots & 0 & 0 \end{pmatrix},$$

These functions satisfy the following PDE system:

$$\begin{aligned} \frac{\partial f_i}{\partial x^j} &= f_i \frac{\partial f_1}{\partial x^{j+1}} + \frac{\partial f_{i+1}}{\partial x^{j+1}}, \\ \frac{\partial f_n}{\partial x^j} &= f_n \frac{\partial f_1}{\partial x^{j+1}}. \end{aligned} \qquad i, j = 1, \dots, n-1. \tag{10}$$

(which is much simpler than Nijenhuis relations but still non-trivial).
So the next step is to solve this system of PDEs.

Solving PDE system (10)

We have already seen that this system of PDEs can be rewritten in the following matrix form:

$$f_{x^j} = L_{\text{comp1}}\, f_{x^{j+1}}, \qquad \text{where} \quad f = \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}$$

or, equivalently,

$$f_{x^{n-k}} = (L_{\text{comp1}})^k f_{x^n}.$$

This is the same kind of system as we treated when proving "Companion Form Theorem". In turn, the latter system is equivalent to

$$\left(\frac{\partial f}{\partial x}\right) L_{\text{comp1}}(f) = L_{\text{comp1}}(f) \left(\frac{\partial f}{\partial x}\right) \tag{11}$$

We first discuss the case when the Jacobi matrix $\left(\frac{\partial f}{\partial x}\right)$ is non-degenerate (equivalently, L_{comp1} is differentially non-degenerate). The trick is to interchange f and x: we will think of $f_1, \dots, f_n$ as local coordinates and $x^i = x^i(f)$ as unknown functions. After this, (11) becomes a linear PDE system (of the exactly same form!):

$$\left(\frac{\partial x}{\partial f}\right) L_{\text{comp1}}(f) = L_{\text{comp1}}(f) \left(\frac{\partial x}{\partial f}\right) \quad \text{or} \quad x_{f^{n-k}} = (L_{\text{comp1}})^k x_{f^n}. \tag{12}$$

Solving PDE system (12)

Proposition 8

For any real analytic initial condition

$$\begin{aligned} x^1(c_1, \ldots, c_{n-1}, c_n + \tau) &= v_1(\tau) \\ &\ldots \\ x^n(c_1, \ldots, c_{n-1}, c_n + \tau) &= v_n(\tau) \end{aligned} \tag{13}$$

where τ belongs to a neignborhood of zero, there exists a unique (local) real analytic solution of $\left(\frac{\partial x}{\partial f}\right) L_{\text{comp1}}(f) = L_{\text{comp1}}(f)\left(\frac{\partial x}{\partial f}\right)$, i.e., (12). This solution can be found by using the following procedure. Consider the real analytic function $F(t)$ constructed from $v_1, \ldots, v_n$ as follows:

$$F(t) = v_1\big(p(t)\big) + t\, v_2\big(p(t)\big) + \cdots + t^{n-1} v_n\big(p(t)\big), \tag{14}$$

where $p(t) = t^n - c_1 t^{n-1} - \cdots - c_{n-1} t - c_n$. Then the solution $x(f)$ satisfying the initial conditions (13) takes the form

$$x(f) = F(L)e_n, \quad \text{where } e_n = (0, \ldots, 0, 1)^\top, \tag{15}$$

i.e., $x(f)$ is the last column of the matrix $F(L)$.

Proof of formula (15)

To prove formula (15), it is sufficient to check two facts:

- Formula (15) gives a solution of (12) for an arbitrary polynomial $F(t)$ (hence for any real analytic function)
- The solution defined by (15) indeed satisfies the required initial conditions.

Since (15) is linear, instead of an arbitrary polynomial $F(t)$ it is sufficient to consider only t^k, $k = 0, 1, 2, \ldots$. We will check this fact by induction:

Lemma 9

Let $x(f) = \begin{pmatrix} x^1(f) \\ \vdots \\ x^n(f) \end{pmatrix}$ *be a solution of* (12), *then* $\tilde{x}(f) = Lx(f)$ *is a solution also, i.e., multiplication by L sends solutions to solutions.*

Proof. It is easy to see that the Jacobi matrix $\left(\frac{\partial \tilde{x}}{\partial f}\right)$ takes the form

$$\left(\frac{\partial \tilde{x}}{\partial f}\right) = L\left(\frac{\partial x}{\partial f}\right) + x^1 \cdot \mathrm{Id}$$

Hence, if $\left(\frac{\partial x}{\partial f}\right)$ commutes with L, then $\left(\frac{\partial \tilde{x}}{\partial f}\right)$ commutes also, i.e. $\tilde{x}(f)$ is a solution of (12), as stated.

Proof of formula (15) (continued)

Now it is easy to see $x(f) = e_n$ is a (constant) solution of (12), then by induction, we have Le_n, L^2e_n, $\ldots$, L^ke_n are all solutions too. Hence, $x(f) = F(L)e_n$ is a solution of (12) for any polynomial (or function) F.

To check the initial conditions, we compute $F(L)e_n$, i.e., the last column of $F(L)$ for $f_i = c_i$, $(i = 1, \ldots, n-1)$, $f_n = c_n + \tau$.
First we notice that the substitution of L (with the above indicated values of f_i) into the polynomial $p(t) = t^n - c_1 t^{n-1} - \cdots - c_{n-1}t - c_n$ gives

$$\begin{aligned} p(L) &= L^n - c_1 L^{n-1} - \cdots - c_{n-1}L - c_n\mathrm{Id} \\ &= L^n - f_1 L^{n-1} - \cdots - f_{n-1}L - (f_n - \tau)\mathrm{Id} = \chi_L(L) + \tau \cdot \mathrm{Id} = \tau \cdot \mathrm{Id}. \end{aligned}$$

Next, $F(L) = \sum v_{n-k}(p(L))L^k = \sum v_{n-k}(\tau \cdot \mathrm{Id})L^k = \sum v_{n-k}(\tau)L^k$.
It remains to notice that the last column of L^k is the $(n-k)$-th basis vector e_{n-k} and therefore the last column of $F(L)$ is

$$F(L)e_n = \sum v_{n-k}(\tau)e_{n-k} = \begin{pmatrix} v_1(\tau) \\ \vdots \\ v_n(\tau) \end{pmatrix}, \text{ as required.}$$

Preliminary conclusion

We can solve (without integration) PDE system (12) (and hence (10)) in the "good" situation. More precisely we know how to find all the Nijenhuis operators of the form

$$L_{\mathrm{comp1}}(x) = \begin{pmatrix} f_1 & 1 & 0 & \dots & 0 \\ f_2 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ f_{n-1} & 0 & \dots & 0 & 1 \\ f_n & 0 & \dots & 0 & 0 \end{pmatrix},$$

in the differentially non-degenerate case when the functions $f_1(x), \dots, f_n(x)$ are functionally independent, i.e., $\det\left(\frac{\partial f}{\partial x}\right) \neq 0$.

All these operators, by an appropriate coordinate transformation, can be reduced to a (unique) canonical form with $f_i(x) = x^i$.

However, in this context, we are mostly interested in "singular" case when $\det\left(\frac{\partial f}{\partial x}\right) \neq 0$, i.e., the characteristic map $\Phi = (f_1, \dots, f_n) : M^n \to \mathbb{R}^n$ has a singular point. So our work is not completed.

General formula for $f_1, \dots, f_n$

Theorem 10

Let $r(\lambda, t) = \lambda^n - v_1(t)\lambda^{n-1} - \dots - v_{n-1}(t)\lambda - v_n(t)$, where $v_i(t)$ are arbitrary real analytic functions. Consider the following matrix relation

$$r(L, M) = 0, \tag{16}$$

where $M = x^1 L^{n-1} + x^2 L^{n-2} + \dots + x^{n-1} L + x^n \,\mathrm{Id}$, and L is a gl-regular $n \times n$ matrix. Then in a neighbourhood of $x = 0$:

- *The coefficients $f_1, \dots, f_n$ of the charactertic polynomial of L can be uniquely recovered from (16) as real analytic functions of $x^1, \dots, x^n$ (by means of Implicit Function Theorem).*
- *The functions $f_1(x), \dots, f_n(x)$ so obtained are solutions of (10) with initial conditions*

$$\begin{aligned} f_1(0, \dots, 0, x^n) &= v_1(x^n), \\ f_2(0, \dots, 0, x^n) &= v_2(x^n), \\ &\dots \\ f_n(0, \dots, 0, x^n) &= v_n(x^n). \end{aligned} \tag{17}$$

Proving this theorem

Consider the eigenvalues λ_i of L_{comp1}. At algebraically generic points (recall that the set of such points is open and dense), λ_i's are smooth and satisfy the following system of PDEs (in the companion coordinate system).

Lemma 11

Assume that $x_0 = (x_0^1, \dots, x_0^n)$ is algebraically generic, then (10) implies that every eigenvalue λ of L, in a neighbourhood of this point, satisfies the following system of PDEs

$$\frac{\partial \lambda}{\partial x^{n-k}} = \lambda^k \frac{\partial \lambda}{\partial x^n}, \quad k = 1, \dots, n-1. \tag{18}$$

Conversely, if we have n functions $\lambda_1, \dots, \lambda_n$ each of which satisfies (18), then the coefficients f_i's of the polynomial $\chi(\lambda) = \prod_{i=1}^n (\lambda - \lambda_i) = \lambda^n - \sum_{i=1}^n f_i \lambda^{n-i}$ satisfy (10).

Remark 12

Below we assume that the eigenvalues are real, but everything can be naturally generalised to the case of complex conjugate eigenvalues.

Proof. We rewrite (10) in matrix form

$$\boxed{F_{x^i} = L_{\text{comp1}}(F)\, F_{x^{i+1}}, \quad F = (f_1, \ldots, f_n)^\top.} \tag{19}$$

Let $\Lambda = (\lambda_1, \ldots, \lambda_n)^\top$ be the roots of the polynomial $\chi(\lambda)$, then we have standard polynomial expressions for its coefficients f_i in terms of Λ. Then (19) can be rewritten as

$$\left(\frac{\partial F}{\partial \Lambda}\right) \Lambda_{x^i} = L_{\text{comp1}}(F) \left(\frac{\partial F}{\partial \Lambda}\right) \Lambda_{x^{i+1}}, \tag{20}$$

where $\left(\frac{\partial F}{\partial \Lambda}\right)$ denotes the Jacobi matrix. We now use the following algebraic identity (verify it!)

$$\left(\frac{\partial F}{\partial \Lambda}\right) \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} = L_{\text{comp1}}(F) \left(\frac{\partial F}{\partial \Lambda}\right) \tag{21}$$

If λ_i's are pairwise distinct, then $\left(\frac{\partial F}{\partial \Lambda}\right)$ is invertible and (20) is equivalent to

$$\Lambda_{x^i} = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \Lambda_{x^{i+1}}, \tag{22}$$

which coincides with (18), as required.

Exercises

Exercise 1

Prove that an operator L_{comp2} in the second companion form (3) is Nijenhuis if and only is relations (4) hold (in other words, verify the second statement of Theorem 4.

Exercise 2

Let $\lambda_1, \ldots, \lambda_n$ be the roots of the polynomial

$$p(\lambda) = \lambda^n - f_1\lambda^{n-1} - \cdots - f_{n-1}\lambda - f_n.$$

Verify the algebraic identity (21).

Exercise 3

Check that the statement of Lemma 11 still holds true for complex eigenvalues $\lambda(x) = \alpha(x) + i\beta(x)$.

Exercise 4

Consider two operators $L_{\text{comp1}} = \begin{pmatrix} x_1 & 1 \\ x_2 & 0 \end{pmatrix}$ and $L_{\text{comp2}} = \begin{pmatrix} 0 & 1 \\ y_2 - \frac{1}{2}y_1^2 & y_1 \end{pmatrix}$.

Make sure that they are both Nijenhuis and find a coordinate change $x = x(y)$ that transform L_{comp1} to L_{comp2}.