Nijenhuis Geometry

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Lecture 14:
$\mathfrak{gl}$-regular Nijenhuis operators (part 3)

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Outline of Lecture 12

- Local description of $\mathfrak{gl}$-regular Nijenhuis operators (not local classification yet)
- Algebraic formulas for solving PDEs (of very special kind)
- Nijenhuis perturbations of a Jordan block
- One example
- Exercises

First companion form

Theorem 1

Let L be a real analytic gl-regular Nijenhuis operator with characteristic polynomial $\chi_L(\lambda) = \det(\lambda \cdot \mathrm{Id} - L) = \lambda^n - f_1\lambda^{n-1} - \dots - f_n$.
Then in a suitable coordinate system $x = (x^1, \dots, x^n)$, this operator takes the form

$$L_{\mathrm{comp1}}(x) = \begin{pmatrix} f_1 & 1 & 0 & \dots & 0 \\ f_2 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ f_{n-1} & 0 & \dots & 0 & 1 \\ f_n & 0 & \dots & 0 & 0 \end{pmatrix}, \tag{1}$$

where $f_i = f_i(x)$ are the coefficients of the characteristic polynomial in this coordinate system. They satisfy the following PDE system:

$$\begin{aligned} \frac{\partial f_i}{\partial x^j} &= f_i \frac{\partial f_1}{\partial x^{j+1}} + \frac{\partial f_{i+1}}{\partial x^{j+1}}, \\ \frac{\partial f_n}{\partial x^j} &= f_n \frac{\partial f_1}{\partial x^{j+1}}. \end{aligned} \qquad i, j = 1, \dots, n-1. \tag{2}$$

How to solve the PDE system (2)?

In the companion coordinate system $x^1, \dots, x^n$, the coefficients of the characteristic polynomial of a Nijenhuis operator $L = L_{\mathrm{comp1}}$ satisfy:

$$\begin{aligned} \frac{\partial f_i}{\partial x^j} &= f_i \frac{\partial f_1}{\partial x^{j+1}} + \frac{\partial f_{i+1}}{\partial x^{j+1}}, \\ \frac{\partial f_n}{\partial x^j} &= f_n \frac{\partial f_1}{\partial x^{j+1}}. \end{aligned} \qquad i, j = 1, \dots, n-1.$$

Conversely, any solution $f_1, \dots f_n$ of this system gives a Nijenhuis operator in companion form L_{comp1}.

This system of PDEs can be rewritten in the following matrix form:

$$f_{x^{n-k}} = (L_{\mathrm{comp1}})^k f_{x^n} \quad \text{or} \quad f_{x^i} = L_{\mathrm{comp1}}\, f_{x^{i+1}}, \qquad \text{where} \quad f = \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} \tag{3}$$

In turn, the latter system is equivalent to

$$\left(\frac{\partial f}{\partial x}\right) L_{\mathrm{comp1}}(f) = L_{\mathrm{comp1}}(f) \left(\frac{\partial f}{\partial x}\right) \tag{4}$$

Some general remarks on system (2) (= (3))

The system

$$\boxed{f_{x^{n-k}} = L^k_{\text{comp1}}(f)\, f_{x^n}.}$$

is a quasilinear (of hydrodynamic type) of the exactly same form as we discussed when proving Companion Form Theorem, i.e., $u_{x^{n-k}} = L^k(u) u_{x^n}$.

We know already that this sort of systems are consistent and admit a local real analytic solution for any initial condition of the form $u(0, \dots, 0, x^n) = v(x^n)$, if $L(u)$ is a Nijenhuis operator.

Question. Is this condition fulfilled in our case? In other words, is $L_{\text{comp1}}(f)$ Nijenhuis?

Answer. Yes, of course. $L_{\text{comp1}}(f) = \begin{pmatrix} f_1 & 1 & & \\ \vdots & 0 & \ddots & \\ f_{n-1} & \vdots & \ddots & 1 \\ f_n & 0 & \dots & 0 \end{pmatrix}$ is Nijenhuis if we think of $f_1, \dots, f_n$ as local coordinates (independent variables).

Conclusion. The system (2) (= (3) = (4)) admits a real analytic solution for any real analytic initial condition $f_i(0, \dots, 0, x^n) = v_i(x^n)$.

Reminder: how to solve the system in the case when $\det\left(\frac{\partial f}{\partial x}\right) \neq 0$

In this case, the map $x = (x^1, \dots, x^n) \mapsto f(x) = (f_1, \dots, f_n)$ is invertible and it will be easier to describe the inverse transformation $x = x(f)$.

Proposition 2

Take an arbitrary matrix function $F(\cdot)$ that is real analytic in a neighbourhood of $L_{\mathrm{comp1}}(c)$ (with fixed initial values $f_1 = c_1, \dots, f_n = c_n$) and let $x = x(f)$ be the last column of the matrix $F\big(L_{\mathrm{comp1}}(f)\big)$.

If the transformation $f \mapsto x(f)$ so obtained is invertible, then the inverse map $f = f(x)$ solves (2). Any invertible solution of (2) can be obtained in this way.

This proposition gives us a "description" of all differentially non-degenerate Nijenhuis operators in first companion form.

However, we want to have a "formula" for all of them, not necessarily differentially non-degenerate.

General formula for $f_1, \dots, f_n$

Theorem 3

Let $r(\lambda, t) = \lambda^n - v_1(t)\lambda^{n-1} - \dots - v_{n-1}(t)\lambda - v_n(t)$, where $v_i(t)$ are arbitrary real analytic functions. Consider the following matrix relation

$$r(L, M) = 0, \tag{5}$$

where $M = x^1 L^{n-1} + x^2 L^{n-2} + \dots + x^{n-1} L + x^n \,\mathrm{Id}$, and L is a gl-regular $n \times n$ matrix. Then in a neighbourhood of $x = 0$:

- *The coefficients $f_1, \dots, f_n$ of the charactertic polynomial of L can be uniquely recovered from (5) as real analytic functions of $x^1, \dots, x^n$ (by means of Implicit Function Theorem).*
- *The functions $f_1(x), \dots, f_n(x)$ so obtained are solutions of (2) with initial conditions*

$$\begin{aligned} f_1(0, \dots, 0, x^n) &= v_1(x^n), \\ f_2(0, \dots, 0, x^n) &= v_2(x^n), \\ &\dots \\ f_n(0, \dots, 0, x^n) &= v_n(x^n). \end{aligned} \tag{6}$$

Proving this theorem

Step 1. Consider the eigenvalues λ_i of L_{comp1}. At algebraically generic points (recall that the set of such points is open and dense), λ_i's are smooth and satisfy the following system of PDEs (in the companion coordinate system).

Lemma 4

Assume that $x_0 = (x_0^1, \dots, x_0^n)$ is algebraically generic, then (2) implies that every eigenvalue λ of L, in a neighbourhood of this point, satisfies the following system of PDEs

$$\frac{\partial \lambda}{\partial x^{n-k}} = \lambda^k \frac{\partial \lambda}{\partial x^n}, \quad k = 1, \dots, n-1. \tag{7}$$

Conversely, if we have n functions $\lambda_1, \dots, \lambda_n$ each of which satisfies (7), then the coefficients f_i's of the polynomial $\chi(\lambda) = \prod_{i=1}^n (\lambda - \lambda_i) = \lambda^n - \sum_{i=1}^n f_i \lambda^{n-i}$ satisfy (2).

Remark. Below we assume that the eigenvalues are real, but everything can be naturally generalised to the case of complex conjugate eigenvalues.

Proof. We rewrite (2) in matrix form

$$\boxed{F_{x^i} = L_{\mathrm{comp1}}(F)\, F_{x^{i+1}}, \quad F = (f_1, \ldots, f_n)^\top.} \tag{8}$$

Let $\Lambda = (\lambda_1, \ldots, \lambda_n)^\top$ be the roots of the polynomial $\chi(\lambda)$, then we have standard polynomial expressions for its coefficients f_i in terms of Λ. Then (8) can be rewritten as

$$\left(\frac{\partial F}{\partial \Lambda}\right) \Lambda_{x^i} = L_{\mathrm{comp1}}(F) \left(\frac{\partial F}{\partial \Lambda}\right) \Lambda_{x^{i+1}}, \tag{9}$$

where $\left(\frac{\partial F}{\partial \Lambda}\right)$ denotes the Jacobi matrix. We now use the following algebraic identity (verify it!)

$$\left(\frac{\partial F}{\partial \Lambda}\right) \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} = L_{\mathrm{comp1}}(F) \left(\frac{\partial F}{\partial \Lambda}\right) \tag{10}$$

If λ_i's are pairwise distinct, then $\left(\frac{\partial F}{\partial \Lambda}\right)$ is invertible and (9) is equivalent to

$$\Lambda_{x^i} = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \Lambda_{x^{i+1}}, \tag{11}$$

which coincides with (7), as required.

Step 2. We next describe solutions of the PDE system (7), i.e., $\frac{\partial\lambda}{\partial x^{n-k}} = \lambda^k \frac{\partial\lambda}{\partial x^n}$, $k = 1, \ldots, n-1$.

Lemma 5

Let $\mu(t)$ be an arbitrary real analytic function. Then the solution $\lambda(x)$ of (7) with initial condition $\lambda(0, 0, \ldots, 0, x^n) = \mu(x^n)$ can be found by resolving the following algebraic relation

$$\lambda = \mu\left(x^1\lambda^{n-1} + x^2\lambda^{n-2} + \cdots + x^{n-1}\lambda + x^n\right). \qquad (12)$$

Every local real analytic solution of (7) can be obtained in this way.

Proof. First check that the implicit solution $\lambda(x)$ of relation (12) satisfies (7). Indeed, differentiating (12) with respect to x^i we get

$$\lambda_{x^i} = \mu' \cdot \left(c\,\lambda_{x^i} + \lambda^{n-i}\right), \quad \text{where} \quad c = \sum_{\alpha=1}^{n-1} x^\alpha (n-\alpha)\lambda^{n-\alpha-1}.$$

Hence, $\qquad \lambda_{x^i}(1 - \mu'\,c) = \mu'\,\lambda^{n-i} \qquad (*)$

and similarly $\qquad \lambda_{x^{i+1}}(1 - \mu'\,c) = \mu'\,\lambda^{n-i-1} \qquad (**)$

Now multiplying (**) by λ and subtracting from (*) we get $(\lambda_{x^i} - \lambda\,\lambda_{x^{i+1}})(1 - \mu'\,c) = 0$. It remains to notice that $c = 0$ on the initial line $x^1 = x^2 = \cdots = x^{n-1} = 0$ and (7) follows.

The fulfilment of the initial condition and uniqueness are straightforward.

We need to show that the solution $f(x) = \big(f_1(x), \dots, f_n(x)\big)$ of (2) with prescribed initial conditions (6) can be obtained by resolving the relation

$$L^n = v_1(M)L^{n-1} + v_2(M)L^{n-2} + \dots + v_{n-1}(M)L + v_n(M), \qquad (13)$$

where $M = x^1 L^{n-1} + \dots x^{n-1}L + x^n \mathrm{Id}$, with respect to the coefficients of the characteristic polynomial of L.
W.l.o.g. we assume that $L = L_{\mathrm{comp1}}(f)$. The r.h.s. of (13) commutes with L and therefore can be uniquely presented as linear combination

$$\sum v_i(M)L^{n-i} = g_1 L^{n-1} + \dots + g_{n-1}L + g_n \,\mathrm{Id}\,. \qquad (14)$$

Moreover, $g_i = g_i(x, f)$ are exactly the entries of the last column of $\sum v_i(M)L^{n-i}$. Thus, relation (13) reads $L^n = \sum_{i=1}^n g_i L^{n-i}$. Comparing with $L^n = \sum_{i=1}^n f_i L^{n-i}$ (Cayley–Hamilton theorem) and using gl-regularity of L we come to the system of n algebraic relations

$$f_i = g_i(x, f). \qquad (15)$$

Notice that $g_i(0, \dots, 0, x^n, f) = v_i(x^n)$, implying $\frac{\partial g_i}{\partial f_\alpha}(0, \dots, 0, x^n, f) = 0$ (as $v_i(x^n)$ does not depend on f_α). This guarantees solvability of (15) by means of Implicit Function Theorem.
The initial conditions are also fulfilled: $f_i(0, \dots, 0, x^n) = g_i(0, \dots, 0, x^n, f) = v_i(x^n)$, as required.

Step 3. It remains to show that the coefficients $f_1, \dots, f_n$ of the characteristic polynomial of L satisfying (13) solve PDE system (2).

Lemma 6

An algebraically generic operator L satisfies (13) *if and only if the eigenvalues $\lambda_1, \dots, \lambda_n$ of L satisfy*

$$\lambda_i = \mu_i\left(x^1\lambda_i^{n-1} + x^2\lambda_i^{n-2} + \dots + x^{n-1}\lambda_i + x^n\right), \quad i = 1, \dots, n. \quad (16)$$

where the functions $\mu_i(t)$ are the roots of the equation

$$\lambda^n - v_1(t)\lambda^{n-1} - v_2(t)\lambda^{n-2} - \dots - v_{n-1}(t)\lambda - v_n(t) = 0. \quad (17)$$

Proof. Assume that a gl-regular operator $L(x)$ satisfies relation (13). Then on the initial line $x(t) = (0, \dots, 0, t)$ we have

$$L^n - v_1(t)L^{n-1} - \dots - v_{n-1}(t)L - v_n(t)\mathrm{Id} = 0.$$

Consider the polynomial equation (17) with coefficients depending on $t \in U = U(0) \in \mathbb{R}$. The multiplicities of its roots depend on t, but since the functions $v_i(t)$ are real analytic, these multiplicities are constant everywhere except for a discrete subset $\mathrm{Sing} \subset U$. So w.l.o.g. we may assume that all points in U, except perhaps for 0, are non-singular.

Then the roots of (17) are defined by real analytic functions in $U \setminus \{0\}$. More precisely, on $U \setminus \{0\}$ there exist s analytic functions (perhaps complex-valued) $\mu_1(t), \dots, \mu_s(t)$ such that

$$\lambda^n - v_1(t)\lambda^{n-1} - v_2(t)\lambda^{n-2} - \dots - v_{n-1}(t)\lambda - v_n(t) = \prod_{i=1}^{s} \big(\lambda - \mu_i(t)\big)^{k_i},$$

where $\mu_i(t) \neq \mu_j(t)$ for $t \in U \setminus \{0\}$. Notice that $\mu_i(t)$ are exactly the eigenvalues of L on the initial line $x(t) = (0, \dots, 0, t)$.
The matrix relation (13) can therefore be rewritten in the form

$$\prod_{i=1}^{s} \big(L - \mu_i(M)\big)^{k_i} = 0$$

for any point $(x^1, \dots, x^n)$ sufficiently close to $(0, \dots, 0, t)$. This, in turn, implies that each eigenvalue $\lambda_\alpha = \lambda_\alpha(x)$ of $L(x)$ satisfies the relation

$$\prod_{i=1}^{s} \big(\lambda_\alpha - \mu_i(x^1\lambda_\alpha^{n-1} + \dots + x^{n-1}\lambda_\alpha + x^n)\big)^{k_i} = 0.$$

Taking into account the fact that the eigenvalues of $L(x)$ depend on x continuously and for $x = (0, \dots, 0, t)$ they are $\mu_i(t)$ with multiplicities k_i, we conclude that at every point $x = (x^1, \dots, x^n)$ sufficiently close to $(0, \dots, 0, t)$, the operator $L(x)$ has s eigenvalues $\lambda_i(x)$ with multiplicities k_i and, moreover, these eigenvalues satisfy (16), as required.

Summarising discussion to complete the proof

We are now ready to complete the proof of Theorem 3.

Let L satisfy (13), then by Lemma 6 its eigenvalues satisfy (16) and therefore (by Lemma 5) are solutions of the PDE system (7). By Lemma 4, the coefficients $f_1, \dots, f_n$ of the characteristic polynomial of $L(x)$ satisfy (2) as required.

In short:

$$(13) \stackrel{\text{Lemma 6}}{\Rightarrow} (16) = (12) \stackrel{\text{Lemma 5}}{\Rightarrow} (7) \stackrel{\text{Lemma 4}}{\Rightarrow} (2)$$

Strictly speaking, this proof works in a small neighbourhood of the set $\{(0, \dots, 0, t),\ t \in U \setminus \{0\}\}$. However, the final conclusion still holds in a neighbourhood of the origin $(0, \dots, 0, 0)$ due to analyticity of f.

This completes the proof of Theorem 3 (that provides algebraic formula for the functions $f_1, \dots, f_n$ from L_{comp1}).

Nijenhuis perturbations of Jordan block

Our next goal is to discuss Nijenhuis perturbations of a Jordan block J_0, that is, Nijenhuis operators of the form

$$L(x) = J_0 \ + \ \text{higher order terms.}$$

Recall that a generic Nijenhuis perturbation of J_0 is described by

Proposition 7

Assume that $L(\mathrm{p}) = J_0$ and the differentials of the coefficients of the characteristic polynomial of L are linearly independent at p. Then in a neighbourhood of p there exist local coordinates $x^1, \dots, x^n$ with

$$\mathrm{p} \simeq (0, \dots, 0) \text{ in which } L(x) = \begin{pmatrix} x^1 & 1 & & \\ \vdots & 0 & \ddots & \\ x^{n-1} & \vdots & \ddots & 1 \\ x^n & 0 & \dots & 0 \end{pmatrix}.$$

Important fact. For any collection of real and complex conjugate numbers $S = \{\lambda_1, \dots, \lambda_k, \mu_1, \bar{\mu}_1, \dots, \mu_s, \bar{\mu}_s\}$ $(k + 2s = n)$ sufficiently close to zero and not necessarily distinct, there exists a unique point $\mathrm{q} \in U(\mathrm{p})$ such that S is the spectrum of $L(\mathrm{q})$. In particular, in $U(\mathrm{p})$ we can find operators of all (potentially admissible) algebraic types.

Problem of a Jordan block

Question. Let L be a Jordan block at a point $\mathrm{p} \in M$. What can we say about behaviour of L in a neighbourhood of p, if L is Nijenhuis? What are possible scenarios? Can we, for instance, "perturb" a Jordan block in such a way that exactly two eigenvalues appear with prescribed multiplicities?

This is a question about solutions of PDE system (2)

$$\frac{\partial f_i}{\partial x^j} = f_i \frac{\partial f_1}{\partial x^{j+1}} + \frac{\partial f_{i+1}}{\partial x^{j+1}}, \quad \frac{\partial f_n}{\partial x^j} = f_n \frac{\partial f_1}{\partial x^{j+1}},$$

that describes the coefficients of the characteristic polynomial

$$\chi_{L(x)}(\lambda) = \lambda^n - f_1(x)\lambda^{n-1} - \dots - f_n(x).$$

We know already that this system is consistent and has solutions for all initial conditions $f_i(0, \dots, 0, x^n) = v_i(x^n)$, $i = 1, \dots, n$.

Problem of the discriminant. Let the characteristic polynomial of $L(x)$ has a multiple root on the initial line $x(t) = (0, \dots, 0, t)$. Is it true that the characteristic polynomial of $L(x)$ has a multiple root for all $x = (x^1, \dots, x^n)$.

A little bit of algebraic geometry

Consider a polynomial with varying coefficients

$$\chi_{(f_1,\ldots,f_n)}(\lambda) = \lambda^n - f_1\lambda^{n-1} - \ldots - f_n.$$

How do its roots and their multiplicities depend on $f_1, \ldots, f_n$?
Need to look at the structure of the discriminant set

$$\{\mathrm{Discr}_\chi = 0\} = \bigsqcup_{\sum k_s = n} S_{k_1,\ldots,k_s} \subset \mathbb{R}^n(f_1, \ldots, f_n).$$

where $S_{k_1,\ldots,k_s}$ is the set of $(f_1, \ldots, f_n) \in \mathbb{R}^n$ for which χ_f has $s < n$ distinct roots with multiplicities $k_1, \ldots, k_s$.

Theorem 8

Let $f_1(x), \ldots, f_n(x)$ be a solution of (2) with initial conditions $f_1(0, \ldots, 0, t) = v_1(t), \ldots, f_n(0, \ldots, 0, t) = v_n(t)$ such that

$$\big(v_1(t), \ldots, v_n(t)\big) \in S_{k_1,\ldots,k_s}.$$

Then

$$(f_1(x), \ldots, f_n(x)) \in \bar{S}_{k_1,\ldots,k_s},$$

for all x.

Perturbation of a Jordan block

Consider the natural stratification of the set of regular operators

$$\mathrm{gl}(n,\mathbb{R})^{\mathrm{reg}} = \bigsqcup_{\sum k_s = n} W_{k_1,\dots,k_s}, \qquad k_1 \le \dots \le k_s,\ s \le n,\ k_i \in \mathbb{N},$$

where $W_{k_1,\dots,k_s}$ is the set of operators having s distinct eigenvalues with multiplicities $k_1,\dots,k_s$. Notice that the Jordan block belongs to the closure of each of $W_{k_1,\dots,k_s}$.

The next theorem states that under (Nijenhuis) perturbation of a Jordan block J_0 all scenarios are allowed.

Theorem 9

For each stratum $W_{k_1,\dots,k_s} \subset \mathrm{gl}(n,\mathbb{R})$, there exists a Nijenhuis operator L in a neighborhood of $0 \in \mathbb{R}^n$ such that $L(0) = J_0$ and $L(x) \in \overline{W}_{k_1,\dots,k_s}$ for all $x \in U(0)$, where $\overline{W}_{k_1,\dots,k_s}$ is the closure of $W_{k_1,\dots,k_s}$ (either in the standard or Zariski topology).

Proof.

To construct the corresponding perturbation one just needs to make sure that the desired scenario happens on the initial line $x(\tau) = (0, \dots, 0, \tau)$. Assume that on this initial line at a generic point $\tau \in (-\varepsilon, \varepsilon)$ we have

$$\begin{aligned} \chi_{L(x(\tau))}(t) &= t^n - v_1(\tau)t^{n-1} - \dots - v_n(\tau) \\ &= (t - \mu_1(\tau))^{k_1}(t - \mu_2(\tau))^{k_2} \cdots (t - \mu_s(\tau))^{k_s}. \end{aligned} \tag{18}$$

where $\mu_i(\tau)$ are some real analytic functions in τ (perhaps complex valued). In other words, generically this polynomial has s distinct roots with multiplicities $k_1, \dots, k_s$.

According to Theorem 3, to describe the solution $f = f(x)$ with given initial conditions $f(x(\tau)) = v(\tau)$ we should consider the relation

$$r(L, M) = L^n - v_1(M)L^{n-1} - \dots - v_n(M) = 0, \quad \text{with} \quad M = \sum_{i=1}^{n} x^i L^{n-i},$$

and then "solve" it to find the coefficients of the characteristic polynomial of L in terms of $x^1, \dots, x^n$. Notice that $r(L, M)$ is just the polynomial (18) after the substitution $\tau \mapsto M$, $t \mapsto L$. Hence we can write

$$r(L, M) = (L - \mu_1(M))^{k_1}(L - \mu_2(M))^{k_2} \cdots (L - \mu_s(M))^{k_s} = 0$$

where μ_i is now treated as an analytic matrix function.

The eigenvalues of L (as functions in x) can now be found from relations of the form:

$$\lambda = \mu_i \left(x_1 \lambda^{n-1} + x_2 \lambda^{n-2} + \cdots + x_{n-1} \lambda + x_n \right)$$

By the Implicit Function Theorem this can be done uniquely in a neighbourhood of a point $(0, \dots, 0, \tau)$ in such a way that $\lambda(0, \dots, 0, \tau) = \mu_i(\tau)$, as needed.
No other eigenvalues may occur. The multiplicities of these eigenvalues will be as expected since this condition is fulfilled on the initial line. This shows that at a generic point we have

$$\chi_{L(x)}(t) = (t - \lambda_1(x))^{k_1} (t - \lambda_2(x))^{k_2} \cdots (t - \lambda_s(x))^{k_s}$$

where $\lambda_i(x)$ will be real analytic functions such that $\lambda_i(0, \dots, 0, \tau) = \mu_i(\tau)$ (these relations hold as soon as $\mu_i(\tau)$ makes sense).

Example

Consider the case of dim = 3 and, in the settings of Theorem 3, define the initial conditions in such a way that on the initial line $x(\tau) = (0, 0, \tau)$ the characteristic polynomial of L takes the form

$$\chi_{L(x(\tau))}(\lambda) = (\lambda - \tau)^2(\lambda - 2\tau) = \lambda^3 - 4\tau\lambda^2 + 5\tau^2\lambda - 2\tau^3,$$

or equivalently

$$\begin{aligned} f_1(0,0,\tau) &= \ 4\tau \ = v_1(\tau), \\ f_2(0,0,\tau) &= -5\tau^2 = v_2(\tau), \\ f_3(0,0,\tau) &= \ 2\tau^3 \ = v_3(\tau). \end{aligned}$$

The algorithm described in Theorem 3 allows us to reconstruct the functions f_1, f_2, f_3. To that end, we use the matrix relation

$$L^3 - (4M)L^2 + (5M^2)L - 2M^3 = 0 \quad \text{with} \quad M = x_1 L^2 + x_2 L + x_3 \,\mathrm{Id},$$

to express the coefficients of the characteristic polynomial of L in terms of x_1, x_2 and x_3.

Notice that this relation can be rewritten as $(L - M)^2(L - 2M) = 0$, which immediately allows us to find the eigenvalues of L by taking the roots of the polynomial

$$(\lambda - x_1\lambda^2 - x_2\lambda - x_3)^2(\lambda - 2x_1\lambda^2 - 2x_2\lambda - 2x_3) = 0$$

Recall that we are interested in specific roots, namely those which, on the initial curve, coincide with the above prescribed roots, that is,

$$\lambda_1(0,0,x_3) = \lambda_2(0,0,x_3) = x_3, \quad \lambda_3(0,0,x_3) = 2x_3.$$

In this particular case we just need to choose the right root (one of the two) of the corresponding quadratic equation. Namely,

$$\lambda - x_1\lambda^2 - x_2\lambda - x_3 = 0 \quad \Rightarrow \quad \lambda = \frac{2x_3}{(1-x_2) + \sqrt{(1-x_2)^2 - 4x_1x_3}}$$

$$\lambda - 2x_1\lambda^2 - 2x_2\lambda - 2x_3 = 0 \quad \Rightarrow \quad \lambda = \frac{4x_3}{(1-2x_2) + \sqrt{(1-2x_2)^2 - 16x_1x_3}}$$

As a result we have found explicit expressions for the eigenvalues of the Nijenhuis operator L in coordinates x_1, x_2, x_3: $\lambda_1 = \lambda_2 = \lambda$, $\lambda_3 = \lambda$.
The final conclusion is:

$$L_{\text{comp1}} = \begin{pmatrix} f_1(x) & 1 & 0 \\ f_2(x) & 0 & 1 \\ f_3(x) & 0 & 0 \end{pmatrix} \quad \text{with} \quad \begin{aligned} f_1 &= \lambda_1 + \lambda_2 + \lambda_3, \\ f_2 &= -\lambda_1\lambda_2 - \lambda_2\lambda_3 - \lambda_3\lambda_1, \\ f_3 &= \lambda_1\lambda_2\lambda_3, \end{aligned}$$

is a Nijenhuis perturbation of the nilpotent 3×3 Jordan block J_0 under which J_0 splits into two Jordan blocks, of size 2 and 1 with non-constant eigenvalues.

Exercises

Exercise 1

Let $L = J_0 +$ (higher order terms) be a Nijenhuis perturbation of the Jordan block. Prove the following criterion for this perturbation to be differentially non-degenerate.
At the initial point p (such that $L(\mathrm{p}) = J_0$) consider a tangent vector $\xi \in T_{\mathrm{p}}M$ that does not belong to the image of J_0 and the function $f_n = \det L$. Then L is differentially non-degenerate at $\mathrm{p} \in M$ if and only if $\xi(f_n) \neq 0$.

Exercise 2

In assumptions of Proposition 2, show that the matrix $F(L_{\mathrm{comp1}})$ can always be reconstructed from its last column $F(L_{\mathrm{comp1}})e_n$. (Hint: if M commutes with a companion matrix L_{comp1}, then M can be uniquely reconstructed from its last column Me_n.) Here $e_n = (0, \dots, 0, 1)^\top$.

Exercise 3

In assumptions of Proposition 2, show that every column of the matrix $F(L_{\mathrm{comp1}})$ gives a solution $x(f)$. More generally, $F(L_{\mathrm{comp1}})\,c$ is a solution for any constant vector $c = (c_1, \dots, c_n)^\top$.