Nijenhuis Geometry

Alexey Bolsinov and Andrey Konyaev

Lecture 15:
gl-regular Nijenhuis operators (part 4)

Sino-Russian Mathematical Center
Moscow University and Peking University

Spring 2021

to contact me: bolsinov@mail.ru

Outline of Lecture 12

- gl-regular operators in dimension 2
- Local classification and normal forms
- Structure of the singular set
- Global examples in dimension 2
- Topological obstructions
- Exercises

Important Notice! All the objects in this lecture are real analytic.
In the smooth case there are still many open questions.

gl-regularity in dimension 2

As we know, gl-regularity of an operator L means that each of its eigenvalues admits only one eigenvector (up to propostionality) or, equivalenlty, only one Jordan block in its Jordan normal form.

In dimension 2, we have linear operators of the following algebraic types:

- two distinct real eigenvalues: $L = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$, $\lambda \neq \mu$;
- two complex conjugate eigenvalues $L = \begin{pmatrix} \lambda & -\mu \\ \mu & \lambda \end{pmatrix}$, $\mu \neq 0$;
- Jordan block: $L = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$;
- scalar matrix: $L = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$.

We see that the first three types are gl-regular, and the only exception are scalar matrices $\lambda \cdot \mathrm{Id}$.

Conclusion. An operator L on a two-dimensional manifold is gl-regular if and only if L admits no singular points of scalar type, i.e., $L(x) \neq \lambda \cdot \mathrm{Id}$ at any point.

Local classification in dimension 2

First three types of non-singular points

- Two real eigenvalues:

$$L = \begin{pmatrix} f(x) & 0 \\ 0 & g(y) \end{pmatrix},$$

where $f(x)$ and $g(y)$ are smooth functions such that $f(x) \neq g(y)$ for all (x, y). In real analytic case, $f(x)$ is either constant of reduces to $f(x) = f_0 \pm x^m$ by coordinate transformation. Similar for $g(y)$.

- Two complex conjugate eigenvalues:

$$L = \begin{pmatrix} f(x,y) & -g(x,y) \\ g(x,y) & f(x,y) \end{pmatrix},$$

where $h = f + \mathrm{i}\, g$ is holomorphic in $z = x + \mathrm{i}\, y$, $g(x, y) \neq 0$ for all (x, y). Here $h(z)$ is either constant or reduces to $h(z) = h_0 + z^m$, $m \in \mathbb{N}$ by a coordinate transformation.

- Jordan block:

$$L = \begin{pmatrix} f(y) & 1 \\ 0 & f(y) \end{pmatrix},$$

where $f(y)$ is a smooth function. In real analytic case, $f(y)$ is either constant or reduces to $f(y) = f_0 \pm y^{2m}$.

Perturbation of a Jordan block in dimension 2

Recall that a point $\mathrm{p} \in \mathrm{M}$ is singular, if the algebraic type of L changes at this point. In dimension 2 under the gl-regularity assumption, there is only one possibility for $L(\mathrm{p})$ (in appropriate coordinates):

$$L(\mathrm{p}) = \lambda \cdot \mathrm{Id} + J_0, \quad \text{where } J_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ \lambda = \mathrm{const} \in \mathbb{R}.$$

Since $L - \lambda \cdot \mathrm{Id}$ is still a Nijenhuis operator, we will assume w.l.o.g. that $L(\mathrm{p}) = J_0$ which lead us to "Nijenhuis perturbation problem" for J_0.

Consider the coefficients of the characteristic polynomial

$$\chi_L(\lambda) = \det(\lambda \cdot \mathrm{Id} - L) = \lambda^2 - v\lambda - u, \quad v = \operatorname{tr} L, \ u = -\det L.$$

Two essentially different cases:

- either $\mathrm{d}\, v \wedge \mathrm{d}\, u \equiv 0$,
- or $\mathrm{d}\, v \wedge \mathrm{d}\, u \neq 0$ on an open dense subset.

In the latter case, L can be reconstructed from v and u (see (8), Lecture 2):

$$L = \begin{pmatrix} v_x & v_y \\ u_x & u_y \end{pmatrix}^{-1} \begin{pmatrix} v & 1 \\ u & 0 \end{pmatrix} \begin{pmatrix} v_x & v_y \\ u_x & u_y \end{pmatrix}, \quad v = \operatorname{tr} L, \ u = -\det L. \qquad (1)$$

For this reason in Theorem 1 below, when appropriate, instead of the matrix of L we will give formulas for $v(x, y)$ and $u(x, y)$.

If $\mathrm{d}\, v \wedge \mathrm{d}\, u \equiv 0$, then (1) makes no sense, but we may still use another more general relation (see (8), Lecture 2):

$$\begin{pmatrix} v_x & v_y \\ u_x & u_y \end{pmatrix} \begin{pmatrix} l_1^1 & l_2^1 \\ l_1^2 & l_2^2 \end{pmatrix} = \begin{pmatrix} v & 1 \\ u & 0 \end{pmatrix} \begin{pmatrix} v_x & v_y \\ u_x & u_y \end{pmatrix}, \qquad (2)$$
$$v = \operatorname{tr} L, \quad u = -\det L, \quad L = \begin{pmatrix} l_1^1 & l_2^1 \\ l_1^2 & l_2^2 \end{pmatrix}.$$

We will assume that L is defined in a neighbourhood of the origin $\mathrm{p} = (0,0) \in \mathbb{R}^2(x,y)$ and coordinate transformations always leave the origin fixed.

The theorem below provides the complete list of normal forms for L.

They are divided into several series.

Local classification in dimension 2 (continued)

Theorem 1

Let L be a Nijenhuis operator such that $L(\mathrm{p}) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then in suitable local coordinates (x, y), this operator takes one of the following forms:

1. *Series L:* $\quad L_{\mathrm{nil}} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad L_{\mathrm{nd}} = \begin{pmatrix} x & 1 \\ y & 0 \end{pmatrix},$

2. *Series M:* $\quad M_{2k-1} = \begin{pmatrix} 0 & 1 \\ 0 & y^{2k-1} \end{pmatrix}, \quad M^{\epsilon}_{2k} = \begin{pmatrix} 0 & 1 \\ 0 & \epsilon y^{2k} \end{pmatrix},$

3. *Series N:* $\quad N_{2k-1} = \begin{pmatrix} y^{2k-1} & 1 \\ 0 & y^{2k-1} \end{pmatrix}, \quad N^{\epsilon}_{2k} = \begin{pmatrix} \epsilon y^{2k} & 1 \\ 0 & \epsilon y^{2k} \end{pmatrix}.$

Here $k \in \mathbb{N}$, $\epsilon = \pm 1$.

(to be continued . . .)

Local classification in dimension 2 (continued)

Important property: L can be uniquely recovered from the coefficients of its characteristic polynomial $v = \operatorname{tr} L$ and $u = -\det L$ as follows:

$$L = \begin{pmatrix} v_x & v_y \\ u_x & u_y \end{pmatrix}^{-1} \begin{pmatrix} v & 1 \\ u & 0 \end{pmatrix} \begin{pmatrix} v_x & v_y \\ u_x & u_y \end{pmatrix}, \qquad v = \operatorname{tr} L,\ u = -\det L.$$

4. Series $O^{d,\epsilon}_{k,\mathrm{c}}$, $k \geq 1$, $d \geq 2k+1$, $\epsilon = \pm 1$, $\mathrm{c} = (c_0, \dots, c_{k-1}) \in \mathbb{R}^k$ and we set $\epsilon = 1$, if $d = 2m+1$ is odd:

$$v = \alpha x y^{2k-1} + y^k\big(c_{k-1}y^{k-1} + \dots + c_1 y + c_0\big), \quad u = \epsilon\, y^d, \quad \alpha = k c_0^2\big(1 - \tfrac{k}{d}\big) \neq 0.$$

5. Series $P^{k,\epsilon}_{s,\mathrm{c}}$, $k \geq 1$, $s \geq 2k$, $\epsilon = \pm 1$, $\mathrm{c} = (c_0, \dots, c_{k-1}) \in \mathbb{R}^k$:

$$v = \alpha x y^s + y^{s-k+1}\big(c_{k-1}y^{k-1} + \dots + c_1 y + c_0\big) + 2\epsilon\, y^k, \quad u = -y^{2k}, \quad \alpha = 2\epsilon\, k c_0 \neq 0.$$

6. Series $S^{2k,\epsilon}_{\mathrm{c}}$ and S^{2k+1}_{c}, $k \geq 1$, $\mathrm{c} = (c_0, \dots, c_{k-1}) \in \mathbb{R}^k$:

$$v = \alpha x y^{2k-1} + y^k\big(c_{k-1}y^{k-1} + \dots + c_1 y + c_0\big), \quad u = \epsilon\, y^{2k}, \quad \alpha = \tfrac{k}{2}\big(c_0^2 + 4\epsilon\big) \neq 0,$$

$$v = \alpha x y^{2k} + y^{k+1}\big(c_{k-1}y^{k-1} + \dots + c_1 y + c_0\big), \quad u = y^{2k+1}, \quad \alpha = 2k+1.$$

Reducing $\det L$ to a nice form

Lemma 2

Under assumptions of Theorem 1, there exist local coordinates (x, y) such that for $u = -\det L$ one of the following holds:

$$\text{(i) } u \equiv 0, \quad \text{(ii) } u = \pm y^{2k}, \quad \text{(iii) } u = y^{2k-1}, \qquad k \in \mathbb{N}.$$

Proof. Let $v = \operatorname{tr} L$, $u = -\det L$. Then in companion coordinates:

$$\begin{pmatrix} v_x \\ u_x \end{pmatrix} = \begin{pmatrix} v & 1 \\ u & 0 \end{pmatrix} \begin{pmatrix} v_y \\ u_y \end{pmatrix} \text{ implying } u_x = g(x,y)u, \text{ where } g(x,y) = v_y.$$

Hence $u = f(y)\exp\left(\int_0^x g(t,y)dt\right)$ for some real analytic function $f(y)$.
If $f(y) \equiv 0$, we have Case (i).
Otherwise, writing f in the form $f(y) = \epsilon y^m h(y)$ with $\epsilon = \pm 1$, $h(0) > 0$, $m \in \mathbb{N}$, we get for $m = 2k$ and $m = 2k-1$ respectively:

$$u = \pm\left(y\sqrt[2k]{h\exp\left(\int_0^x g(t,y)dt\right)}\right)^{2k} \text{ or } u = \left(y\sqrt[2k-1]{\pm h\exp\left(\int_0^x g(t,y)dt\right)}\right)^{2k-1}.$$

Letting y_{new} be the expression in brackets gives $u = \pm y_{\text{new}}^{2k}$ or $u = y_{\text{new}}^{2k-1}$, as required.

Step 2. Simplify L keeping the y-coordinate unchanged

Lemma 3

There exists a coordinate change of the form $(x_{\mathrm{old}}, y) \mapsto (x, y)$ *such that the* l_2^1*-component of* L *in new coordinates equals identically* 1.

Proof. Setting $x_{\mathrm{old}} = g(x, y)$ and applying the standard transformation rule for components of an operator:

$$\begin{pmatrix} * & 1 \\ * & * \end{pmatrix} = L_{\mathrm{new}} = C^{-1} L_{\mathrm{old}} C = \begin{pmatrix} g_x^{-1} & -g_y g_x^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} l_1^1 & l_2^1 \\ l_1^2 & l_2^2 \end{pmatrix} \begin{pmatrix} g_x & g_y \\ 0 & 1 \end{pmatrix},$$

we observe that the required condition is

$$\frac{l_2^1(g, y) + g_y l_1^1(g, y) - g_y l_2^2(g, y) - g_y^2 l_1^2(g, y)}{g_x} = 1,$$

where l_j^i are the components of L in the old coordinate system.

Writing this relation in the form $g_x = F(g_y, g, y)$, we can locally solve it by Cauchy-Kovalevskaya theorem.

Important: Since L is gl-regular, we can choose initial conditions in such a way that $g_x(0, 0) \neq 0$ so that the coordinate transformation is invertible.

Case by case...

Now let us discuss all the cases one by one.

- First, assume that $u \equiv 0$ and $v \equiv 0$, then L is a nilpotent Jordan block and its companion form coincides with L_{nil}.
- Next suppose $u \equiv 0$, while v is not. In companion coordinates, we have

$$\begin{pmatrix} v_x \\ u_x \end{pmatrix} = \begin{pmatrix} v & 1 \\ u & 0 \end{pmatrix} \begin{pmatrix} v_y \\ u_y \end{pmatrix} \text{ with } u \equiv 0 \text{ implying } v_x = v v_y.$$

This equation can be rewritten as $v_x = g(x, y)v$ with $g = v_x$.

Just in the same way as in Lemma 2, we find a coordinate system in which $v = y^{2k-1}$ or $v = \epsilon y^{2k}$ for $k \geq 1, \epsilon = \pm 1$.

By Lemma 3, we may also assume that $l_2^1 = 1$. Now $L = \left(l_j^i\right)$ can be reconstructed from relation (2):

$$\begin{pmatrix} 0 & v_y \\ 0 & 0 \end{pmatrix} \begin{pmatrix} l_1^1 & 1 \\ l_1^2 & l_2^2 \end{pmatrix} = \begin{pmatrix} v & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & v_y \\ 0 & 0 \end{pmatrix} \quad \Rightarrow \quad l_1^2 = 0,\ l_2^2 = v.$$

This yields series M_{2k-1} and M_{2k}^{ϵ} for different v respectively.

Case by case (continued)

- Now let $u \not\equiv 0$, but $\mathrm{d}\,v \wedge \mathrm{d}\,u \equiv 0$. Combining Lemmas 2 and 3, we may assume that $u = y^{2m-1}$ or $u = \pm y^{2m}$ for $m \geq 1$, and $l_2^1 = 1$. Since $\mathrm{d}\,v \wedge \mathrm{d}\,u \equiv 0$, we also know that $v_x \equiv 0$. Relation (2) gives

$$\begin{pmatrix} 0 & v_y \\ 0 & u_y \end{pmatrix} \begin{pmatrix} l_1^1 & 1 \\ l_1^2 & l_2^2 \end{pmatrix} = \begin{pmatrix} v & 1 \\ u & 0 \end{pmatrix} \begin{pmatrix} 0 & v_y \\ 0 & u_y \end{pmatrix} \quad \Rightarrow \quad l_1^2 = 0.$$

  and we come to the operator of the form

$$L = \begin{pmatrix} f(y) & 1 \\ 0 & g(y) \end{pmatrix} \quad \text{with } v = f + g \text{ and } u = -fg.$$

  Furthermore, the Nijenhuis condition gives $f_y'(f - g) = 0$.

  In our case f cannot be constant (otherwise, since L is nilpotent at the origin, we would necessarily have $f \equiv 0$, which contradicts our assumption that $u = -fg \not\equiv 0$).

  Hence, $f - g = 0$, meaning that L is a Jordan block at each point any yielding series N_{2k-1} and N_{2k}^{ϵ}.

- If $\mathrm{d}\,v \wedge \mathrm{d}\,u \neq 0$ at the point p, then L is differentially non-degenerate and its normal form is L_{nd}.

Most complicated case

Finally we consider the most interesting case when $\mathrm{d}\,v \wedge \mathrm{d}\,u \not\equiv 0$ (but $\mathrm{d}\,v \wedge \mathrm{d}\,u = 0$ at p). As previously, we assume that $u = y^{2m+1}$ or $u = \epsilon y^{2m}$ for $m \geq 1, \epsilon = \pm 1$ and $l_2^1 = 1$. Computing l_2^1 from matrix relation (1)

$$\begin{pmatrix} * & 1 \\ * & * \end{pmatrix} = L = \begin{pmatrix} v_x & v_y \\ 0 & u_y \end{pmatrix}^{-1} \begin{pmatrix} v & 1 \\ u & 0 \end{pmatrix} \begin{pmatrix} v_x & v_y \\ 0 & u_y \end{pmatrix},$$

yields the following equation on v:

$$v_x = vv_y - \frac{1}{d} y (v_y)^2 + u_y, \tag{3}$$

where $d = 2m+1$ or $2m$. This equation implies the following

Lemma 4

The function $v(x, y)$ can be written as $v = v_0(y) + y^s(\alpha x + F)$, where $\alpha \neq 0$, $s \geq 1$ and $F(x, y)$ is a real analytic function with no constant or linear part.

Using this Lemma we will change $x_{\mathrm{old}} \mapsto x_{\mathrm{new}}$ to simplify $v = \operatorname{tr} L$ (keeping $u = -\det L$ unchanged).

Proof. Let $v(x,y)=v_0(y)+v_1(y)x+v_2(y)x^2+\dots$ be a solution of (3). This expansion is uniquely determined by the initial condition $v_0(y)$.

Notice that $v_1(y)$ cannot be identically zero (otherwise all terms vanish and $v(x,y)=v_0(y)$ contradicting the condition $\mathrm{d}\,v\wedge\mathrm{d}\,u\not\equiv 0$).
Thus, in our case $v_1(y)=y^s r_1(y)$, where $r_1(0)=\alpha\neq 0$.

Next, $s\geq 1$ (otherwise $s=0$ and $\mathrm{d}\,v$ and $\mathrm{d}\,y$ are linearly independent at p; setting $x_{\mathrm{new}}=v$ one can show that p is differentially non-degenerate, which is not the case).
Now, equating the coefficients of x^i it the both sides of (3) yields

$$(i+1)v_{i+1}(y)=v_0(y)v_i'(y)+v_0'(y)v_i(y)-\frac{2}{d}yv_0'(y)v_i'(y)+ \\ +\sum_{j=1}^{i-1}\left(v_j(y)v_{i-j}'(y)-\frac{1}{d}yv_j'(y)v_{i-j}'(y)\right). \tag{4}$$

If $v_1,\dots,v_i$ are divisible by y^s, then $v_1',\dots,v_i'$ are divisible by y^{s-1}. As $v(0,0)=0$, then v_0 is divisible by y. By formula (4) the coefficient v_{i+1} is divisible by y^s. Thus, by induction all the coefficients $v_1,v_2,\dots$ are divisible by y^s and we have $v=v_0+y^s(xr_1(y)+\dots)=v_0+y^s(\alpha x+F)$, where F is analytic and has no constant or linear parts. Lemma is proved.

Using $v = \operatorname{tr} L = v_0(y) + y^s(\alpha x + F)$, we introduce new coordinates

$$x_{\text{new}} = x + \frac{1}{\alpha} F + \widetilde{v}_0, \quad y_{\text{new}} = y,$$

where $\widetilde{v}_0$ contains all the terms of v_0 of order $\geq s+1$. In this new coordinate system (we keep notation x and y for new coordinates):

$$v = p_s(y) + \alpha x y^s, \quad u = y^{2m+1} \text{ or } u = \epsilon y^{2m}, \tag{5}$$

where $\alpha \neq 0$, $m, s \geq 1$ and p_s is polynomial of degree at most s.

This coordinate system is optimal in the sense that $v = \operatorname{tr} L$ and $u = -\det L$ cannot be simplified further. It remains to decide which polynomials p_s are admissible.

Straightforward reconstruction of L, from (1) with v and u given by (5) gives

$$L = \begin{pmatrix} v - \frac{y v_y}{d} & \frac{v v_y - \frac{1}{d} y v_y^2 + u'}{v_x} \\ \frac{y v_x}{d} & \frac{y v_y}{d} \end{pmatrix}$$

where $d = 2m+1$ or $d = 2m$ (power of y in the formula for u).

The "troublesome" component, in more detail, reads:

$$l_2^1 = s\alpha x^2 y^{s-1}\left(1-\frac{s}{d}\right)+x\left(\frac{1}{y}p_s+\left(1-\frac{2}{d}\right)p_s'\right)+\frac{p_s p_s' - \frac{1}{d}y(p_s')^2 + u'}{\alpha y^s}.$$

Notice that $p_s(0) = 0$ and therefore $\frac{1}{y}p_s$ is analytic. Hence, we only need to analyse the fraction

$$\frac{p_s p_s' - \frac{1}{d}y(p_s')^2 + u'}{\alpha y^s} \tag{6}$$

This fraction must be an analytic function having value 1 at the origin (in order for $L(\mathrm{p})$ to be the standard nilpotent Jordan block). Thus, we need to solve

Purely algebraic problem. Find all polynomials p_s, $\deg p_s \leq s$, for which the denominator of (6) is divisible by αy^s so that this fraction is, in fact, a polynomial with free term equal to 1.

Solving this problem for all $u = \pm y^d$ leads to the series O, P and S, completing the proof of the classification Theorem.

Uniqueness

For the series O, P and S the canonical coordinate system is essentially unique.

Indeed, these coordinates are those in which $u = -\det L$ and $v = \operatorname{tr} L$ are given by explicit formulas (5):

$$v = p_s(y) + \alpha x y^s, \quad u = y^{2m+1} \text{ or } u = \epsilon y^{2m},$$

The integer parameters m and s involved in (5) are uniquely defined for given u and v. Hence, y can be reconstructed from u (sometimes up to sign), and x is determined, up to a constant factor, by the condition that $v(0, y)$ is a polynomial of degree $\leq s$. Finally, the rescaling of x is chosen in such a way that at the origin we have $L(0, 0) = J_0$.

This implies that Nijenhuis operators from different series (or from the same series but with different parameters) are not equivalent to each other. The only exception is related to the above mentioned "canonical" transformation $(x, y) \mapsto (-x, -y)$ that changes the parameter $\mathrm{c} \in \mathbb{R}^k$, but this change is easy to control.

Singular set

Let (M^2, L) be a gl-regular Nijenhuis manifold of dimension 2.
Let Sing be the set of singular points of L where the algebraic type of L changes. In our case, this means that the eigenvalues of L collide, i.e.

$$\mathrm{Sing} = \{\mathrm{p} \in \mathrm{M}^2 \mid v^2 + 4u = 0\}, \quad \text{where } v = \operatorname{tr} L,\ u = -\det L,$$

unless $v^2 + 4u \equiv 0$ on M^2 meaning that L is similar to a Jordan block at each point.
From Theorem 1 we immediately obtain a *local* description of Sing in canonical coordinates x, y:

- for L_{nil}, N_{2k-1} and N^{ϵ}_{2k}, the singular set is empty;
- for L_{nd} the singular set is $\mathrm{Sing} = \{x^2 + 4y = 0\}$;
- for all the other series M, O, P and S: $\mathrm{Sing}_{loc} = \{y = 0\}$.

Thus, locally Sing is a smooth curve. Since $\mathrm{Sing} \subset \mathrm{M}^2$ is closed, we may think of it as a submanifold consisting, perhaps, of several connected components:

$$\mathrm{Sing} = \mathrm{S}_1 \cup \cdots \cup \mathrm{S}_\ell.$$

If M is compact, then each of them is an embedded circle.

Singular set

Next we can easily observe that all points from S_i relate to the same series (different components may, of course, relate to different series). However, the parameters of the series may change. This happens for series O, P and S. Indeed, moving along $\mathrm{Sing}_{loc} = \{y = 0\}$ leads to the shift $x_{\mathrm{new}} = x - x_0$ resulting in the following modification for $v = \mathrm{tr}\, L$:

$$\begin{aligned} v = \alpha x y^s + c_{k-1} y^s + \ldots &= \alpha(x_{\mathrm{new}} + x_0) y^s + c_{k-1} y^s + \ldots \\ &= \alpha x_{\mathrm{new}} y^s + (c_{k-1} + \alpha x_0) y^s + \ldots \end{aligned}$$

Thus, c_{k-1} undergoes the shift $c_{k-1} \mapsto c_{k-1} + \alpha x_0$ so that if we move along S_i in a certain direction, then c_{k-1} is either strictly increasing or strictly decreasing. Important conclusion: singular points from the series O, P and S may not occur on closed gl-regular Nijenhuis 2-manifolds.

Proposition 5

Let (M^2, L) be a closed gl-regular Nijenhuis 2-manifold. Then

- either Sing is empty (i.e. all points of M^2 are of the same algebraic type),
- or each $p \in \mathrm{Sing}$ belongs to the series M and then automatically one of the eigenvalues of L is constant on M^2.

Global example: two-dimensional torus T^2

Example 6

- Two constant eigenvalues λ_1 and λ_2. Let ξ and η be two vector fields on T^2, that are linearly independent at each point. Then we define L as follows:

$$L(\xi) = \lambda_1 \xi \quad \text{and} \quad L(\eta) = \lambda_2 \eta. \tag{7}$$

- One of eigenvalues is constant (and equals zero), the other is not. In usual angle coordinates (ϕ_1, ϕ_2), we set

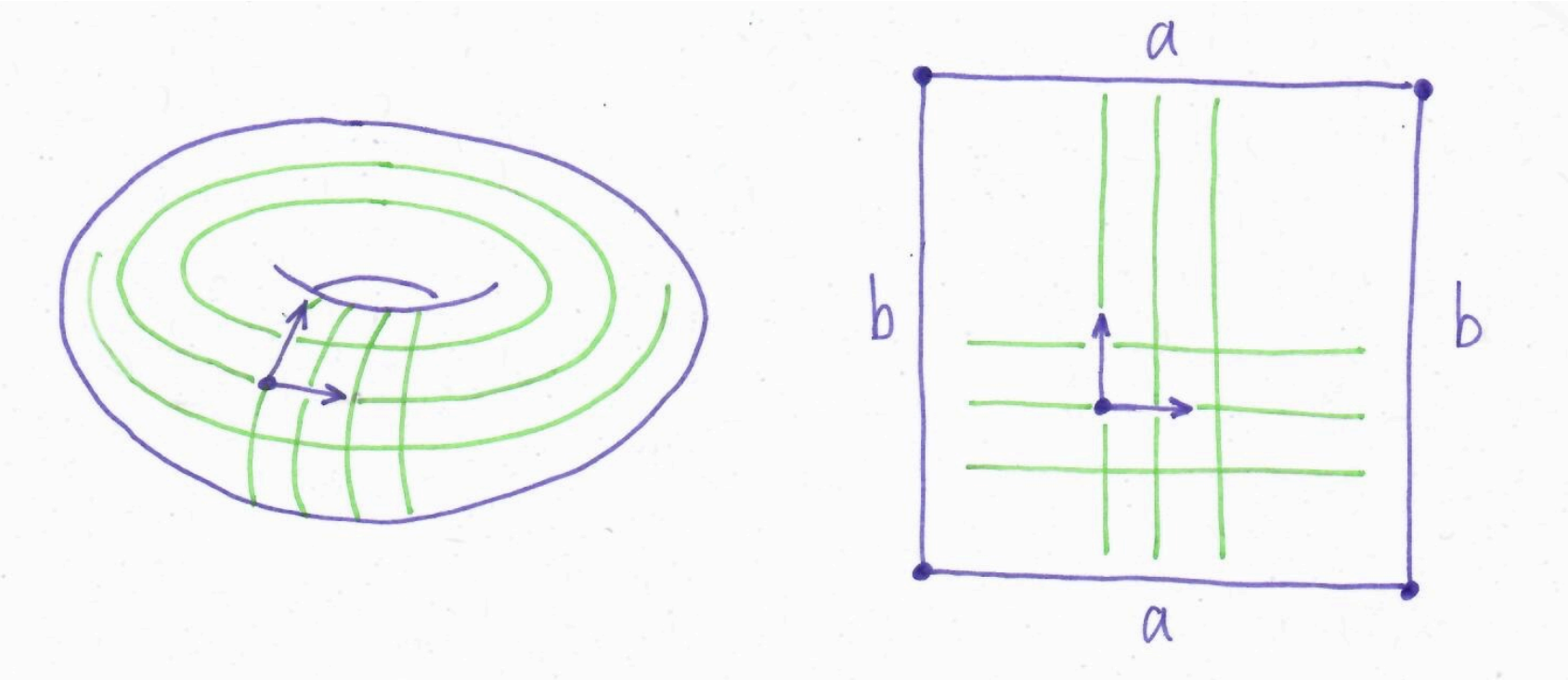
$$L = \begin{pmatrix} 0 & g(\phi_1, \phi_2) \\ 0 & f(\phi_2) \end{pmatrix}. \tag{8}$$

- Two non-constant eigenvalues λ_1 and λ_2. Obvious example:

$$L = \begin{pmatrix} f(\phi_1) & 0 \\ 0 & g(\phi_2) \end{pmatrix}, \qquad f(\phi_1) < c < g(\phi_2). \tag{9}$$

This example can be generalised by taking finite covering over this "standard" torus. On the covering torus such a *global diagonalisation* is not always possible.

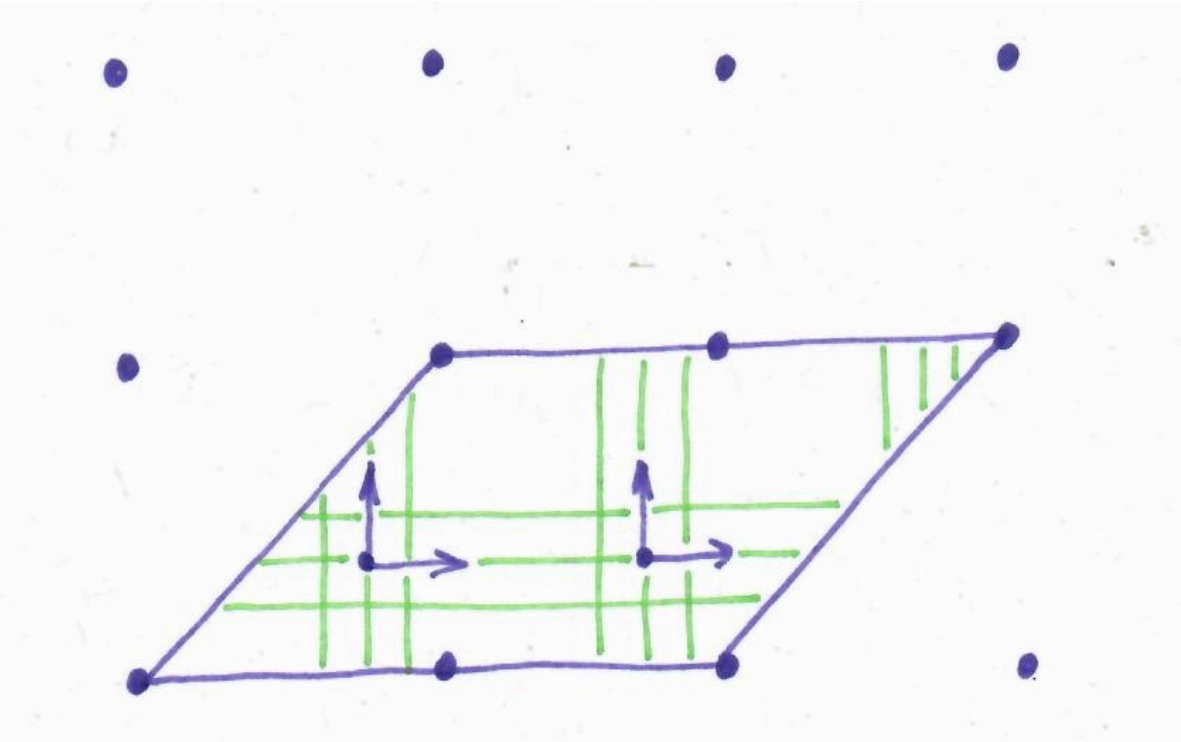
Illustration 1



$$L = \begin{pmatrix} \sin\phi_1 & 0 \\ 0 & 2+\sin\phi_2 \end{pmatrix}$$

$$\mathsf{T}^2 = \mathbb{R}/\Gamma, \quad \Gamma = 2\pi\mathbb{Z} \times 2\pi\mathbb{Z}$$

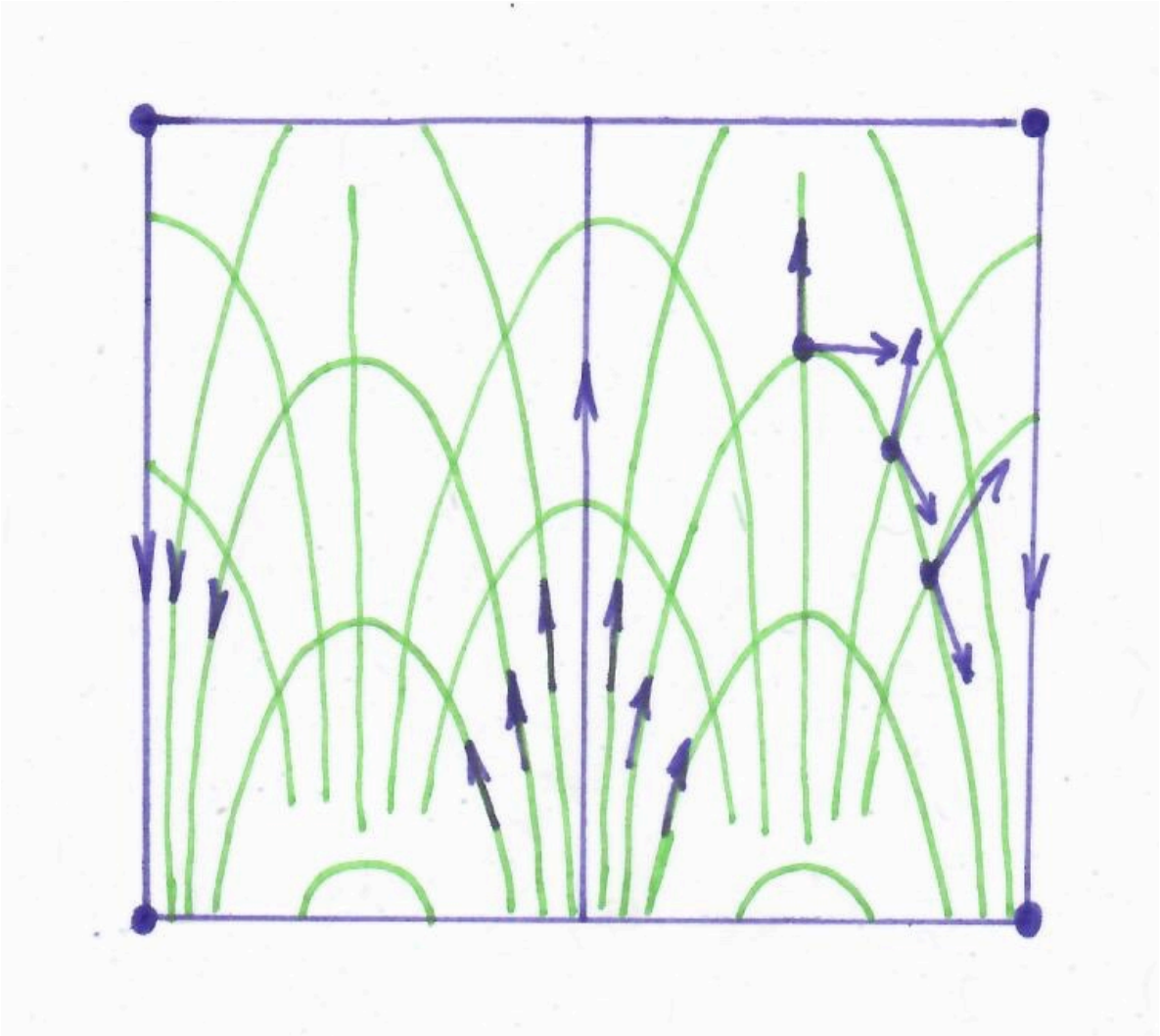
Illustration 2



$$L = \begin{pmatrix} \sin\phi_1 & 0 \\ 0 & 2+\sin\phi_2 \end{pmatrix}$$

$$\mathsf{T}^2 = \mathbb{R}/\Gamma, \quad \Gamma = \mathrm{Span}_{\mathbb{Z}}(2e_1, e_1+e_2)$$
$$= \{(2k+m)e_1 + me_2, \ k.m \in \mathbb{Z}\}$$

Illustration 3



An example of two linearly independent vector fields on the torus

Global examples: Klein bottle

Example 7

Each of the operators (7), (8) and (9) can be naturally "transferred" to the Klein bottle K^2 viewed as quotient of T^2 by involution $\sigma : \mathsf{T}^2 \to \mathsf{T}^2$, where $(\phi_1, \phi_2) \overset{\sigma}{\mapsto} (-\phi_1, \phi_2 + \pi)$. We only need L to be invariant w.r.t. σ.

Therefore, in addition in each of the above cases we need to assume that

- ξ is σ-invariant, and η change orientaion under the action of σ, i.e. $\mathrm{d}\,\sigma(\xi) = \xi$ and $\mathrm{d}\,\sigma(\eta) = -\eta$,
- $f(\phi_2)$ is π-periodic, and $g(\phi_1, \phi_2)$ is even in variable ϕ_1,
- $g(\phi_2)$ is π-periodic, and $f(\phi_1)$ is even.

If these conditions are fulfilled then operators L given by (7), (8) and (9) naturally descend to the quotient $\mathsf{K}^2 = \mathsf{T}^2/\sigma$, and therefore define gl-regular operators on the Klein bottle.

Global examples: torus T^2 again

Example 8

Assume that L is a gl-regular operator on T^2 with double eigenvalue λ. If $\lambda = \mathrm{const}$, then we may set $\lambda = 0$, i.e. think of L as a nilpotent operator.

- Consider two vector fields ξ and η on T^2 that are linearly independent at each point and define L as follows:

$$L(\xi) = 0, \quad L(\eta) = \xi.$$

  Then L is a gl-regular nilpotent Nijenhuis operator on T^2 (notice that every nilpotent operator in dimension 2 is Nijenhuis automatically).

- The case with a non-constant eigenvalue can be modelled on T^2 as follows:

$$\begin{pmatrix} f(\phi_2) & g(\phi_1, \phi_2) \\ 0 & f(\phi_2) \end{pmatrix}, \qquad g(\phi_1, \phi_2) > 0,$$

  where ϕ_1, ϕ_2 are standard angle coordinates on the torus.

Global results in dimension 2

Theorem 9

Let (M^2, L) *be a closed connected* gl*-regular Nijenhuis manifold. Then we have one of the following possibilities:*

1. M^2 *is orientable and* $L = \alpha\,\mathrm{Id} + \beta J$*, where* J *is a complex structure on* M^2 *and* $\alpha, \beta \in \mathbb{R}$ *are constants,* $\beta \neq 0$*.*
2. M^2 *is homeomorphic to either torus or Klein bottle, and* L *has two distinct real eigenvalues at each point of* M^2*.*
3. M^2 *is homeomorphic to a torus, and* L *is similar to a Jordan block at each point of* M^2*.*
4. M^2 *is homeomorphic to either torus or Klein bottle, and one of eigenvalues of* L *is constant.*

Corollary 10

1. *Let* M^2 *be orientable and* $\mathrm{M}^2 \not\simeq \mathrm{T}^2$*. Then on* M^2 *there are no* gl*-regular Nijenhuis operators except for* $L = \alpha\,\mathrm{Id} + \beta J$*.*

2. *Let* M^2 *be non-orientable and* $\mathrm{M}^2 \not\simeq \mathrm{K}^2$*. Then on* M^2 *there are no* gl*-regular Nijenhuis operators at all.*

Sketch of the proof

Consider the two options from Proposition 5. First assume that Sing $= \emptyset$. Then L belongs to one of three generic types listed in the beginning of this Section:

(i) either L has two distinct real eigenvalues at each point of M^2;

(ii) or L has two complex conjugate eigenvalues at each point of M^2;

(iii) or L is similar to a Jordan block at each point of M^2.

In Case (i), at each point $p \in M^2$, we have an eigenbasis basis $e_1, e_2 \in T_p M^2$ where e_1 corresponds to the maximal eigenvalue at a given point. If we fix some Riemannian metric on M^2, we may assume that e_i are normalised so that $|e_i| = 1$. Since such e_i are defined up to $\pm$, we have 4 different bases at each point.

A priori it is not clear whether or not we can chose a smooth "moving frame" field on the whole manifold, but this can obviously be done on a finite sheeted covering $\widetilde{M}^2$ of M (number of sheets is at most four). This implies that $\widetilde{M}^2$ is parallelisable and hence is a torus. Therefore, M^2 is either a torus or Klein bottle and we obtain Case 2 of Theorem 9.

In Case (ii), the complex eigenvalues $\lambda, \bar{\lambda}$ of the Nijenhuis operator L are constant and we obtain Case 1 from Theorem 9.

In Case (iii), at each point $p \in M^2$ we have a non-zero eigenvector $e \in T_p M^2$ and the same argument as above shows that on M^2 or on its two sheeted covering one can define a smooth vector field with no singular points. Hence M^2 is either a torus or Klein bottle. However, in this case we have one additional property that the automorphism group of a Jordan block consists of orientation preserving transformations, which allows us to define orientation on M^2. Hence, the Klein bottle is forbidden and we are lead to Case 3 of Theorem 9.

Thus, the condition $\mathrm{Sing} = \emptyset$ necessarily implies one of the first three cases of Theorem 9.

Finally, we consider the second option from Proposition 5. This option implies that one of the eigenvalues of L is constant allowing us to consider a non-zero eigenvector related to this eigenvalue at each point and, in the same way as above, to construct a smooth vector field with no zeros either on M^2 or on its two-sheeted covering. This implies that M^2 is either a torus or Klein bottle and we obtain Case 4 of Theorem 9.

Thus, the list of possibilities presented in Theorem 9 is complete.

Exercises

Exercise 1

Let L be a Nijenhuis operator in dimension 2. Assume that $L(x) \in \mathrm{sl}(2, \mathbb{R})$ at each point x. Prove that the eigenvalues of L are constant. Is this still true in arbitrary dimension?

Exercise 2

Construct an example of a Nijenhuis operator L in $\mathbb{R}^2$ such that $L(x) \equiv 0$ outside a unit disc, but inside this disc there are some points where the eigenvalues of L are distinct, non-constant and real.

Exercise 3

Prove the following fact (which we used when proving Theorem 9): If M^2 (not necessarily compact) admits an operator L (not necessarily Nijenhuis) which is similar to a Jordan block at each point $x \in \mathrm{M}$, then M orientable.

Prove that this property holds true in any even dimension.

Exercise 4

Does there exist a gl-regular operator L (in dimension 2) such that at some point $x \in \mathrm{M}^2$ we have $\mathrm{d} \det L(x) \neq 0$, $\mathrm{d}\,\mathrm{tr}\, L(x) = 0$?