Nijenhuis Geometry

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Lecture 1:
Introduction. What is Nijenhuis Geometry?

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Outline of the course

- Nijenhuis torsion and Nijenhuis operators: equivalent definitions.
- Basic properties of Nijenhuis operators. Splitting theorem.
- Generalised Nirenberg-Newlander theorem.
- Normal forms for Nijenhuis operators.
- Singular points of Nijenhuis operators and linearisation.
- Left-symmetric algebras. Linearisability and non-degenaracy.
- gl-regular Nijenhuis operators and their canonical forms.
- Normal forms for gl-regular Nijenhuis operators in dimension 2.
- Global properties of Nijenhuis operators on closed manifolds.
- Frölicher-Nijenhuis brackets and Nijenhuis cohomologies.
- Nijenhuis operators and bi-Hamiltonian systems.
- Nijenhuis operators and quasilinear PDEs.
- Niejnhuis operators and Poisson brackets of hydrodynamic type.
- Open problems in Nijenhuis Geometry.

Albert Nijenhuis

Albert Nijenhuis (November 21, 1926 – February 13, 2015),

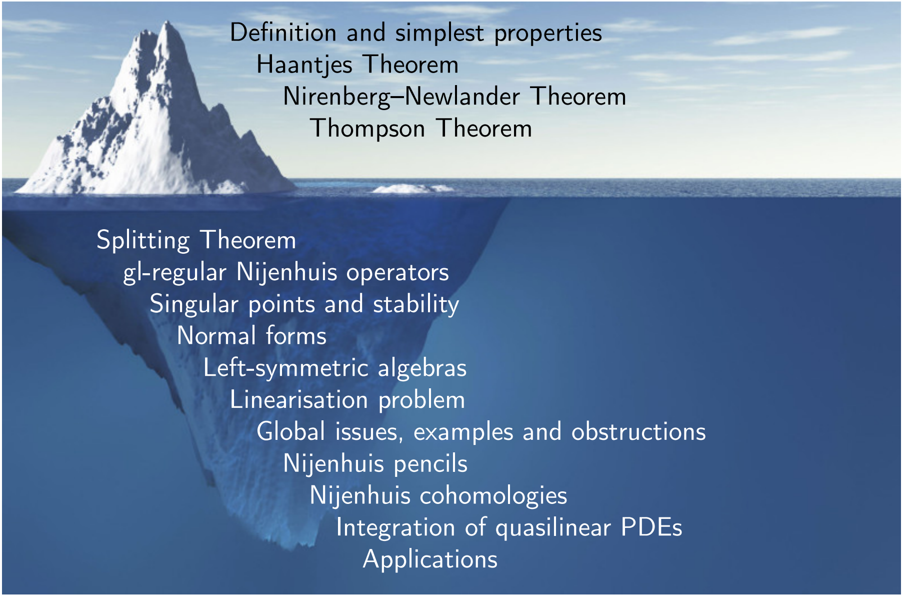
Dutch-American mathematician who specialised in differential geometry and the theory of deformations in algebra and geometry, and later worked in combinatorics.

Alma mater: University of Amsterdam

Doctoral advisor: Prof. Jan Arnoldus Schouten

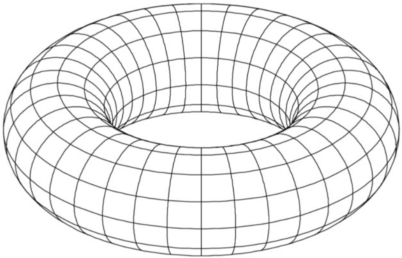
https://en.wikipedia.org/wiki/Albert_Nijenhuis

Nijenhuis Geometry



What is GEOMETRY?

Space, manifold M^n $\quad + \quad$ Structure



Structure is usually defined by means of a tensor, like, g_{ij}, ω_{ij}, or p^{ij}

Naively, in coordinates, the geometric structure is defined by means of a matrix $A = \big(a_{ij}(x)\big)$ whose entries depend on coordinates $x = (x^1, \ldots, x^n)$ and satisfy some algebraic and differential conditions.

Riemannian geometry

Geometric structure is defined by means of a Riemannian metric $g_{ij}(x)$, tensor of type $(0, 2)$, satisfying two algebraic conditions

- symmetry: $g_{ij}(x) = g_{ji}(x)$;
- positive definiteness: $\sum_{i,j} g_{ij}\xi^i\xi^j > 0$ for all $\xi = (\xi^1, \dots, \xi^n) \neq 0$.

Equivalently, we may say that on the tangent space T_xM of every point $x \in M$ we have a bilinear form (with matrix g_{ij}) that defines positive definite inner product on T_xM.

Of course, the matrix $G = \left(g_{ij}(x)\right)$ depends on the choice of local coordinates. Under a coordinate transform $(x^1, \dots, x^n) \mapsto (y^1, \dots, y^n)$, we have

$$G \quad \mapsto \quad J^\top G J$$

where $J = \left(\frac{\partial x^i}{\partial y^j}\right)$ is the Jacobi matrix of this transform.

Differential conditions: flat Riemannian metrics, Ricci flat, Einstein metrics, conformally flat metrics, etc.
Main object here is the Riemann curvature tensor:

$$R(\xi, \eta)\zeta = \nabla_\xi \nabla_\eta \zeta - \nabla_\eta \nabla_\xi \zeta - \nabla_{[\xi,\eta]}\zeta$$

Symplectic geometry

Symplectic structure is defined by means a differential form $\Omega = \big(\omega_{ij}(x)\big)$, tensor of type $(0,2)$ satisfying two algebraic and one differential conditions:

- skew symmetry $\quad \omega_{ij}(x) = -\omega_{ji}(x)$;
- non-degeneracy $\quad \det \Omega \neq 0$;
- closedness $\quad d\Omega = 0$, or $\frac{\partial \omega_{ij}}{\partial x^k} + \frac{\partial \omega_{jk}}{\partial x^i} + \frac{\partial \omega_{ki}}{\partial x^j} = 0$ for all i, j, k.

Equivalently, we may say that on the tangent space $T_x M$ of every point $x \in M$ we have a skew-symmetric non-degenerate bilinear (symplectic) form that is, in addition, closed.
Of course, the matrix $\Omega = \big(\omega_{ij}(x)\big)$ depends on the choice of local coordinates. Under a coordinate transform $(x^1, \dots, x^n) \mapsto (y^1, \dots, y^n)$, we have

$$\Omega \quad \mapsto \quad J^\top \Omega J$$

where $J = \left(\frac{\partial x^i}{\partial y^j}\right)$ is the Jacobi matrix of this transform.
Important fact: locally one can always find a canonical coordinate system such that Ω becomes constant, more precisely $\Omega = \begin{pmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{pmatrix}$.

Poisson geometry

Poisson structure is defined by means of a bivector $P = \left(p^{ij}(x)\right)$, tensor of type $(2,0)$, satisfying one algebraic and one differential conditions:

- skew symmetry $\quad p^{ij}(x) = -p^{ji}(x)$;
- Jacobi identity: if we define Poisson bracket by $\{f,g\} = \sum_{i,j} p^{ij} \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j}$, then for all functions f, g and h:

$$\{\{f,g\},h\} + \{\{g,h\},f\} + \{\{h,f\},g\} = 0$$

Equivalently, on the co-tangent space $T_x^* M$ of every point $x \in M$ we have a skew-symmetric bilinear form (+ Jacobi identity).

The matrix $P = \left(p^{ij}(x)\right)$ depends on the choice of local coordinates.

Under a coordinate transform $(x^1, \ldots, x^n) \mapsto (y^1, \ldots, y^n)$, we have $P \quad \mapsto \quad J^{-1} P (J^{-1})^\top$, where $J = \left(\frac{\partial x^i}{\partial y^j}\right)$ as before.

Splitting theorem: locally one can always find a canonical coordinate system (p^i, q^i, z^α) such that P takes the form $P = \begin{pmatrix} 0 & \mathrm{Id} & 0 \\ -\mathrm{Id} & 0 & 0 \\ 0 & 0 & Q(z) \end{pmatrix}$, and $Q(0) = 0$.

To summarise

We have geometries defined by the following types of "matrices" (tensors)

symmetric bilinear forms on $T_x M$	$\mapsto$	Riemannian Geometry
skew-symmetric bilinear forms on $T_x M$	$\mapsto$	Symplectic Geometry
skew-symmetric bilinear forms on $T_x^* M$	$\mapsto$	Poisson Geometry
symmetric bilinear forms on $T_x^* M$	$\mapsto$	sub-Riemannian Geometry

Differential conditions.

- Vanishing of the Riemann curvature tensor:

$$\nabla_\xi \nabla_\eta - \nabla_\eta \nabla_\xi - \nabla_{[\xi,\eta]} = 0$$

- Closedness of Ω:

$$\xi\, \Omega(\eta, \zeta) + \Omega(\xi, [\eta, \zeta]) + (\text{cyclic permutation}) = 0$$

- Jacobi identity:

$$\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0$$

These are tensors of type $(0, 2)$ and $(2, 0)$. Why not $(1, 1)$?
Why Linear Operators are missing in the list?

Nijenhuis geometry

Geometric structure is defined by means of a linear operator $L = \left(L^i_j(x)\right)$, tensor of type $(1,1)$, satisfying one differential conditions:

- Nijenhuis identity:

$$L^2[\xi,\eta] - L[L\xi,\eta] - L[\xi,L\eta] + [L\xi,L\eta] = 0. \qquad (1)$$

Equivalently, on the tangent space T_xM of every point $x \in M$ we have an operator $L : T_xM \to T_xM$ with vanishing Nijenhuis torsion. Such an operator (tensor field) is called a Nijenhuis operator. The left hand side of (1) is called the Niejnhuis torsion of L and denoted by $\mathcal{N}_L(\xi,\eta)$.

Algebraic conditions will also appear later (e.g., algebraically generic, gl-regular, with complex eigenvalues, nilpotent).

The matrix $L = \left(L^{ij}(x)\right)$ depends on the choice of local coordinates. Under a coordinate transform $(x^1,\ldots,x^n) \mapsto (y^1,\ldots,y^n)$, we have

$$L \quad \mapsto \quad J^{-1}\,L\,J,$$

where $J = \left(\frac{\partial x^i}{\partial y^j}\right)$ as before.
Splitting theorem, canonical coordinates, etc. will follows.

Elementary examples

- Constant operator:
$$L(x) = \left(L^i_j\right)$$
with L^i_j being constant for all i, j

- Scalar operator:
$$L(x) = f(x) \cdot \mathrm{Id},$$
where $f(x)$ is an arbitrary smooth function

- $L(x) = \begin{pmatrix} x_1 & & & \\ & x_2 & & \\ & & \ddots & \\ & & & x_n \end{pmatrix}$

- $L(x) = \begin{pmatrix} x_5 & x_4 & x_3 & x_2 & x_1 \\ & x_5 & x_4 & x_3 & x_2 \\ & & x_5 & x_4 & x_3 \\ & & & x_5 & x_4 \\ & & & & x_5 \end{pmatrix}$

Why is Nijenhuis geometry important? Where and how do Nijenhuis operators appear?

- Nijenhuis Geometry fills a gap in the above list. We will observe many interesting phenomena and results.
- Systems of quasilinear PDEs of the form:

$$u_t = L(u)u_x, \quad u = \begin{pmatrix} u_1(x,t) \\ \vdots \\ u_n(x,t) \end{pmatrix}, \ L(u) \text{ is } n \times n \text{ matrix.}$$

  Under a transformation $u = u(v)$, this systems preserve this form, but $L(u)$ changes by $L \mapsto J^{-1}LJ$ with $J = \left(\frac{\partial u_i}{\partial v_j}\right)$, i.e. L is a $(1,1)$-tensor in the space of 'dependent variables'.
- More complicated geometric structures. Assume that we have two geometric structures from the classical list, e.g., two metrics g and $\bar{g}$ or two Poisson structures P and $\bar{P}$. They are related by means of an operator $\bar{g}(\xi,\eta) = g(L\xi,\eta)$ and similarly $\bar{P}(df,dg) = P(L^*df,dg)$. If g and $\bar{g}$ or P and $\bar{P}$ satisfy certain mutual differential conditions, then very often L turns out to be a Nijenhuis operator.

Nijenhuis relation as compatibility condition

Important fact: Nijenhuis condition is the most natural and simplest differential condition for a linear operator.

Consider a tensor $L(x)$ of type $(1,1)$ (not necessarily Nijenhuis).
Question. What kind of other tensors one can construct form its components L^i_j and their derivatives $\frac{\partial L^i_j}{\partial x^k}$?
Answer. That is the Nijenhuis torsion. The others will be more complicated (e.g. Haantjes tensor) or almost trivial like $d\,\mathrm{tr}\,L$.

Imagine that we have a system of geometric PDEs defined by means of an operator $L^i_j(x)$. Then the compatibility condition for this system will be a geometric condition on L^i_j and $\frac{\partial L^i_j}{\partial x^k}$ (i.e. invariant w.r.t. coordinate transformations). What could this condition be? What about, e.g., $\sum_i \frac{\partial L^i_j}{\partial x^i} = 0$ or something similar? NO, that is impossible. Not geometric!

Conclusion/principle/expectation: If it is not too complicated …it has to be a Nijenhuis condition.

Complex structure

Let $L^i_j(x)$ be an operator on a manifold M^{2n} satisfying the algebraic property $L^2 = -\mathrm{Id}$. Such an L is called an almost complex structure.

If we want L to be a true complex structure, we need to find coordinates $u^1, \dots, u^n, u^{n+1}, \dots, u^{2n}$ (functions on our manifold) such that

$$L(x) = J^{-1} \begin{pmatrix} 0 & -\mathrm{Id} \\ \mathrm{Id} & 0 \end{pmatrix} J, \quad \text{where } J = \left(\frac{\partial u^i}{\partial x^j}\right),$$

or

$$J\, L(x) = \begin{pmatrix} 0 & -\mathrm{Id} \\ \mathrm{Id} & 0 \end{pmatrix} J$$

which is an overdetermined system of linear PDEs.
There should be some non-trivial compatibility conditions that can be formulated in terms of L.

Theorem (Newlander-Nirenberg)

An almost complex structure L is a complex structure if and only if L is Nijenhuis.

Projective geometry

Definition

Two Riemannian metrics g and $\bar{g}$ are projectively equivalent if they have the same geodesics (considered as unparametrised curves).

These two metrics are defined by means of an operator L

$$L = \left(\frac{\det \bar{g}}{\det g}\right)^{\frac{1}{n+1}} \bar{g}^{-1} g.$$

In terms of L, the condition that g and $\bar{g}$ are projectively equivalent takes the following form

$$\nabla_\xi L = \frac{1}{2}\Big(\xi \otimes d\,\mathrm{tr}\,L + (\xi \otimes d\,\mathrm{tr}\,L)^*\Big).$$

This is a system of PDEs on L. However, if we assume that L is given, then this is a system of PDEs on the components $g_{ij}(x)$ of the metric g.

Theorem

If g and $\bar{g}$ are projectively equivalent, then L is Nijenhuis.

Compatible Poisson brackets

Definition

Two Poisson structures P and $\bar{P}$ are called compatible, if the sum $P + \bar{P}$ is a Poisson structure too.

These Poisson structures are related by means of a linear operator L (usually called a recursion operator)

$$L = \bar{P}P^{-1} \quad \text{or, equivalently,} \quad \bar{P}(df, dg) = P(L^* df, dg).$$

Here we assume that P is invertible.

The compatibility condition for P and $\bar{P} = LP$ can be rewritten in terms of P and L as a big system of PDEs.

Theorem (F. Magri)

If P and $\bar{P}$ are compatible, then the recursion operator L is Nijenhuis.

In both cases, projectively equivalent metrics and compatible Poisson brackets, it might be more convenient to replace $(g, \bar{g})$ and $(P, \bar{P})$ with (g, L) and (P, L) respectively.

Left-symmetric algebras

These are generalisation of associative algebras, also known as Koszul-Vinberg or pre-Lie algebras.

Consider an algebra $(\mathfrak{a}, *)$ and ask the following natural question. Under which conditions the operation

$$[a, b] = a * b - b * a$$

satisfies the Jacobi identity so that $(\mathfrak{a}, [\ ,\ ])$ becomes a Lie algebra? The condition is $L_{[a,b]} = L_a \circ L_b - L_b \circ L_a$ or, equivalently,

$$(a * b) * c - a * (b * c) = (b * a) * c - b * (a * c). \qquad (2)$$

Definition

An algebra $(\mathfrak{a}, *)$ is called left-symmetric, if (2) holds.

What is the relation with Nijenhuis geometry?

Recall that the structure constants a^i_{jk} of an algebra $(\mathfrak{a}, *)$ w.r.t. a basis $e_1, \dots, e_n$ are defined by the relation $e_i * e_j = \sum_k a^k_{ij} e_k$.

Theorem

The operator $L = \left(L^i_j(x)\right)$ with $L^i_j = \sum_k a^i_{jk} x^k$ is a Nijenhuis operator if and only if a^i_{jk} are structure constants of a left-symmetric algebra.

Nijenhus torsion (tensor) and Nijenhuis operators

Let $L = (L^i_j(x))$ be an operator (field of endomorphisms, tensor of type $(1,1)$) on a manifold M.

Definition. The Nijenhuis torsion of L is defined by

$$\mathcal{N}_L(\xi,\eta) = L^2[\xi,\eta] - L[L\xi,\eta] - L[\xi,L\eta] + [L\xi,L\eta], \tag{3}$$

where ξ and η are arbitrary vector fields on M.

Proposition. $\mathcal{N}_L$ is a tensor of type $(1,2)$.

Notice that L is skew-symmetric w.r.t. ξ and η (i.e., w.r.t. lower indices). Equivalently, we may say that $\mathcal{N}_L$ is a vector valued 2-form.

Proof. We need to check that, in coordinates, $\big(\mathcal{N}_L(\xi,\eta)\big)^k = N^k_{ij}\xi^i\eta^j$. In other words, there are no partial derivatives of ξ^i and η^j in the right hand side of (3), i.e., all of them cancel out.
Equivalently, we need to show bi-linearity over $C^\infty(M)$, that is,

$$\mathcal{N}_L(\xi, f_1\,\eta_1 + f_2\,\eta_2) = f_1\,\mathcal{N}_L(\xi,\eta_1) + f_2\,\mathcal{N}_L(\xi,\eta_2)$$

for any functions f_1, f_2, and similar for ξ.

Nijenhus torsion is a tensor of type $(1, 2)$ (proof)

For constant coefficients f_1 and f_2, this is obviously true. And we only need to show that ‘a function can be factored out’, that is,

$$\mathcal{N}_L(\xi, f\eta) = f\mathcal{N}_L(\xi, \eta) .$$

Notice that for each term of (3) separately, this condition fails! It holds true only for their linear combination with appropriately chosen signs.

Below we use the property of the Lie bracket: $[\xi, f\eta] = f[\xi, \eta] + \xi(f)\eta]$, where $\xi(f)$ denotes the directional derivative of f along ξ.

$$\mathcal{N}_L(\xi, f\eta) = L^2[\xi, f\eta] - L[L\xi, f\eta] - L[\xi, Lf\eta] + [L\xi, Lf\eta] =$$

$$L^2[\xi, f\eta] - L[L\xi, f\eta] - L[\xi, f\,L\eta] + [L\xi, f\,L\eta] =$$

$$L^2\Big(f[\xi,\eta]+\xi(f)\eta\Big)-L\Big(f[L\xi,\eta]+L\xi(f)\eta\Big)-L\Big(f[\xi,L\eta]+\xi(f)L\eta\Big)+\Big(f[L\xi,L\eta]+L\xi(f)L\eta\Big)$$

$$= f(L^2[\xi,\eta] - L[L\xi,\eta] - L[\xi,L\eta] + [L\xi,L\eta])$$
$$+\,\xi(f)L^2\eta - L\xi(f)L\eta - \xi(f)L^2\eta + L\xi(f)L\eta =$$

$$f(L^2(f[\xi,\eta] - L[L\xi,\eta] - L[\xi,L\eta] + [L\xi,L\eta]) = f\,\mathcal{N}_L(\xi,\eta),$$

as required.

Homework

Exercise

Which of the following operators on $\mathbb{R}^2(x, y)$ are Nijenhuis:

$$\text{a)} \begin{pmatrix} x & y \\ -y & x \end{pmatrix}, \quad \text{b)} \begin{pmatrix} x^2 + y^2 & 0 \\ 0 & x^2 + y^2 \end{pmatrix}, \quad \text{c)} \begin{pmatrix} x & 1 \\ 0 & x \end{pmatrix}, \quad \text{d)} \begin{pmatrix} y & 1 \\ 0 & y \end{pmatrix},$$

$$\text{e)} \begin{pmatrix} x^2 & 0 \\ 0 & y^2 \end{pmatrix}, \quad \text{f)} \begin{pmatrix} x^2 + y^2 & 0 \\ 0 & x^2 - y^2 \end{pmatrix}, \quad \text{g)} \begin{pmatrix} y & x \\ -x & y \end{pmatrix}, \quad \text{h)} \begin{pmatrix} 0 & x \\ x & 2y \end{pmatrix}.$$

Exercise

a) Consider an operator $L = \begin{pmatrix} x & f(x, y) \\ 0 & y \end{pmatrix}$. What are necessary and sufficient conditions for $f(x, y)$ in order for L to be Nijenhuis?

b) Consider an operator $L = \begin{pmatrix} f(x, y) & -g(x, y) \\ g(x, y) & f(x, y) \end{pmatrix}$. What are necessary and sufficient conditions for $f(x, y)$ and $g(x, y)$ in order for L to be Nijenhuis?

Exercise

Let $L = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$ in Cartesian coordinates (x, y). What form does L take in a new coordinate system (u, v), where $u = x + y, v = -xy$.

List of notions/formulas/notations to be used

- Tangent vectors and vector fields ξ, η
- Differential $\mathrm{d}f$ of a function f
- Lie bracket (commutator) of vector fields $[\xi, \eta]$ and Lie derivative $\mathcal{L}_\xi$
- Standard formulas such as
  - $[\xi, \eta] = \mathcal{L}_\xi \eta$,
  - $\mathcal{L}_\xi f = \xi(f) = \mathrm{d}f(\xi) = \langle \mathrm{d}f, \xi \rangle$
  - $\mathcal{L}_\xi (\operatorname{tr} L) = \operatorname{tr}(\mathcal{L}_\xi L)$
  - $\mathcal{L}_\xi (L_1 L_2) = L_1 (\mathcal{L}_\xi L_2) + (\mathcal{L}_\xi L_1) L_2$,
  - $\mathcal{L}_\xi L^{-1} = -L^{-1} (\mathcal{L}_\xi L) L^{-1}$,
  - ...
- Differential 1- and 2-forms, exterior derivative $\alpha \mapsto \mathrm{d}\alpha$
- Operator $L : T_x M \to T_x M$ and its dual operator $L^* : T^*_x M \to T^*_x M$
- Characteristic polynomial, eigenvalues and eigenvectors, invariant subspaces
- Functions of linear operators, like L^k, $\exp(L)$, etc.
- Distribution and integrable distribution, foliations