Nijenhuis Geometry

Alexey Bolsinov and Andrey Konyaev

Lecture 20:
Applications of Nijenhuis Geometry:
bi-Hamiltonian structures and their singularities

Sino-Russian Mathematical Center
Moscow University and Peking University

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to contact me: bolsinov@mail.ru

Outline of Lecture 20

- Poisson structures and Hamiltonian systems
- Compatible Poisson structures, bi-Hamiltonian systems
- Recursion operator and its properties
- Poisson-Nijenhuis structures
- Local normal form for PN structures (algebraically generic case)
- Local normal form for PN structures (singular points)

Poisson structures and Hamiltonian systems

Definition 1

A *Poisson structure* is defined to be a skew-symmetric tensor P^{ij} of type $(2,0)$ such that the corresponding operation (*Poisson bracket*)

$$\{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) \;\to\; C^\infty(M),$$

$$f, g \;\mapsto\; \{f,g\} = P(\mathrm{d}\, f, \mathrm{d}\, g) = \sum P^{ij} \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j}$$

satisfies the Jacobi identity:

$$\{\{f,g\},h\} + \{\{g,h\},f\} + \{\{h,f\},g\} = 0 \quad \text{for all } f,g,h \in C^\infty(M).$$

Remark. If $\det P \neq 0$, then we can define the inverse tensor $\omega = P^{-1}$ which is a differential 2-form. The Jacobi identity, in this case, means exactly that $\mathrm{d}\,\omega = 0$, i.e., ω is a symplectic structure.

Definition 2

Let H be a smooth function. The vector field

$$\mathcal{X}_H = P(\mathrm{d}\, H, \cdot), \quad (\text{equivalently } \mathcal{X}_H^j = P^{ij} \tfrac{\partial H}{\partial x^i} \text{ or } \mathcal{X}_H = P\,\mathrm{d}\, H)$$

is said to be *Hamiltonian* (w.r.t. P and with the *Hamiltonian function* H).

Compatible Poisson structures and bi-Hamiltonian systems

Definition 3
Two Poisson structures P and $\widetilde{P}$ are called *compatible*, if their sum $P + \widetilde{P}$ is also a Poisson structure.

Definition 4
A vector field $\mathcal{X}$ is called *bi-Hamiltonian*, if it is Hamiltonian with respect to two compatible Poisson brackets P and $\tilde{P}$, that is,

$$\mathcal{X} = P\,\mathrm{d}\,H = \widetilde{P}\,\mathrm{d}\,\widetilde{H}.$$

Question. Why are we interested in bi-Hamiltonian systems? What are their properties? And why are such systems "better" than the others?

We will assume here that at least one of the compatible Poisson structures P and $\widetilde{P}$ is non-degenerate so that e.g. P^{-1} makes sense.

Basic properties

Consider the linear operator $L = \widetilde{P} P^{-1}$ or, equivalently,

$$P(L^* \mathrm{d} f, \mathrm{d} g) = \tilde{P}(\mathrm{d} f, \mathrm{d} g) \quad \text{for all } f, g \in C^\infty(M).$$

Proposition 5

Let P and $\widetilde{P}$ be Poisson structures. Then P and $\widetilde{P}$ are compatible if and only if L is Nijenhuis.

In the theory of bi-Hamiltonian systems, the operator L plays an important role and is known as *recursion operator*.

In particular, notice the following formula for Hamiltonian vector fields:

$$\widetilde{\mathcal{X}}_f = L \, \mathcal{X}_f \, .$$

Also notice that without loss of generality we may, at least locally, assume that L is non-degenerate. Indeed, replacing $\tilde{P}$ with $\widetilde{P} + \lambda P$ leads to $L \mapsto L + \lambda \, \mathrm{Id}$.

Proof

We first notice that the Jacobi identity for a Poisson structure P can be rewritten, in terms of Hamiltonian vector fields, as follows:

$$\mathcal{X}_{\{f,g\}} = [\mathcal{X}_f, \mathcal{X}_g].$$

Hence, the Jacobi identity for $P + \widetilde{P}$ can be rewritten as follows (below we use the fact that P and $\widetilde{P}$ are Poisson and remove the terms that vanish automatically due to this fact):

$$\widetilde{\mathcal{X}}_{\{f,g\}} + \mathcal{X}_{\widetilde{\{f,g\}}} = [\widetilde{\mathcal{X}_f}, \mathcal{X}_g] + [\mathcal{X}_f, \widetilde{\mathcal{X}_g}]$$

Now we use the fact that $\widetilde{\mathcal{X}_f} = L\mathcal{X}_f$ for any f:

$$L\mathcal{X}_{\{f,g\}} + L^{-1}\widetilde{\mathcal{X}}_{\widetilde{\{f,g\}}} = [L\mathcal{X}_f, \mathcal{X}_g] + [\mathcal{X}_f, L\mathcal{X}_g]$$

Finally we use again the fact that P and $\widetilde{P}$ are both Poisson to get

$$L[\mathcal{X}_f, \mathcal{X}_g] + L^{-1}[L\mathcal{X}_f, L\mathcal{X}_g] = [L\mathcal{X}_f, \mathcal{X}_g] + [\mathcal{X}_f, L\mathcal{X}_g]$$

Multiplying this identity by L we get the Nijenhuis relation for L. It remains to recall that P is non-degenerate so that Hamiltonian vector fields generate the whole tangent space.

Basic facts (continued...)

Proposition 6

Let $\mathcal{X}$ be a bi-Hamiltonian vector field w.r.t. P and $\widetilde{P}$. Then the coefficients σ_k of the characteristic polynomial of the recursion operator $L = \tilde{P}P^{-1}$ are first integrals of $\mathcal{X}$. Moreover, these integrals commute with respect to the both structures P and $\widetilde{P}$.

Proof. 1. First integrals. It is a well known fact that P is preserved by each Hamiltonian vector field $\mathcal{X}_f$, i.e., $\mathcal{L}_{\mathcal{X}_f} P = 0$. Since in our case $\mathcal{X}$ is bi-Hamiltonian, we conclude that $\mathcal{X}$ preserves both P and $\widetilde{P}$. Hence, $\mathcal{X}$ also preserves the recursion operator $L = \widetilde{P}P^{-1}$, and therefore all of its algebraic invariants such as coefficients σ_k of $\chi_L(t)$.
2. Commutativity. Instead of σ_k, we check commutativity of $f_k = \frac{1}{k}\operatorname{tr} L^k$, i.e., the identities $\{f_k, f_m\} = 0$. We use the following property of Nijenhuis operators $\mathrm{d}\, f_k = L^* \mathrm{d}\, f_{k-1}$ which implies:

$$\{f_k, f_m\} = P(\mathrm{d}\, f_k, \mathrm{d}\, f_m) = P(L^* \mathrm{d}\, f_{k-1}, \mathrm{d}\, f_m) = \{\widetilde{f_{k-1}, f_m}\} = \{f_{k-1}, f_{m+1}\}$$

and by induction $\{f_k, f_m\} = \{\widetilde{f_{k-s}, f_{m+s-1}}\} = \{f_{k-s}, f_{m+s}\}$. It remains to notice that for a suitable s we have either $k-s = m+s$ or $k-s = m+s-1$ and the statement follows from skew symmetry of P and $\tilde{P}$.

Poisson-Nijenhuis structures

Since $\widetilde{P}$ can be recovered from P and L, we can reformulate the compatibility condition for P and $\widetilde{P}$ in terms of L and P only as follows: We will say that a Poisson structure P and Nijenhuis operator L are *compatible* (or define a *Poisson-Nijenhuis structure*) if

(I) the form $\widetilde{P}(\cdot,\cdot) = P(L^* \cdot, \cdot)$ is skew-symmetric, i.e., is a bivector,

(II) $\widetilde{P}$ is Poisson (i.e. satisfies Jacobi identity).

For some reason, it will be more convenient to work with the symplectic forms $\omega = P^{-1}$ and $\widetilde{\omega} = \widetilde{P}^{-1}$ (assuming that P and $\widetilde{P}$ are non-degenerate). As above, we will say that a symplectic structure ω and a Nijenhuis operator L are *compatible* if [1]

(I') the form $\widetilde{\omega}(\cdot,\cdot) = \omega(L\cdot, \cdot)$ is skew-symmetric, i.e., is a differential 2-form,

(II') $\mathrm{d}\,\widetilde{\omega} = 0$, i.e., this form is closed.

(I') and (II') are equivalent to the fact that P and $\widetilde{P}$ are Poisson and compatible.

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[1] This operator L is not the same as before. They are related by inversion, but we prefer to keep the previous notation.

Local nornal forms for Poisson-Nijenhuis structures

It is natural to ask about local simultaneous canonical form for compatible ω and L. In the case when L is diagonalisable and algebraically generic at a point $\mathrm{p} \in M^{2n}$, i.e., its algebraic type remains the same in a certain neighbourhood $U(\mathrm{p})$, the answer is easy to derive from Splitting Theorem.

Notice, first of all, that condition (I') imposes natural algebraic restrictions on the algebraic type of L:

Algebraic fact. If ω and ωL are both skew-symmetric, then in the Jordan decomposition of L all the blocks can be partitioned into pairs of equal blocks (i.e. of the same size and with the same eigenvalue).

In particular, the characteristic polynomial of L is a full square and each eigenvalue has even multiplicity.

The most generic (and simplest) case

Assume that a symplectic structure ω and a Nijenhuis operator L are compatible.

Theorem 7

If L has $n = \frac{1}{2}\dim M$ distinct real eigenvalues, each of multiplicity 2, and in addition their differentials are linearly independent at a point $\mathrm{p} \in M$, then there exists a local symplectic coordinate system $x_1, \dots, x_n,\ p_1, \dots, p_n$ in which L is diagonal with x_i being its i-th eigenvalue:

$$\omega = \sum_i dp_i \wedge dx_i, \quad L = \mathrm{diag}(x_1, x_2, \dots, x_n,\ x_1, x_2, \dots, x_n). \qquad (1)$$

Equivalently, one can say that the pair (ω, L) is the direct product of two-dimensional blocks (ω_i, L_i) of the form $\omega_i = dp_i \wedge dx_i$, $L_i = x_i \cdot \mathrm{Id}$.

Proof

Step 1. Apply the Splitting Theorem:

$$L = L_1 \oplus L_2 \oplus \cdots \oplus L_s = \bigoplus_i L_i$$

Step 2. Use the following algebraic fact.
Let $L = \bigoplus_i L_i$ be a block-diagonal matrix such that L_i and L_j have no common eigenvalues. Let ω be a non-degenerate skew-symmetric matrix such that ωL is skew symmetric too. Then ω is block diagonal, i.e., $\omega = \bigoplus_i \omega_i$.

Step 3. Let ω be a symplectic form such that in some coordinate coordinate system its matrix is block diagonal as above. Then each ω_i is also symplectic and depends on 'its own coordinates' only. In other words, $\omega = \bigoplus_i \omega_i$ is a direct product of symplectic forms.

Step 4. This shows that the Poisson-Nijenhuis structure (ω, L) splits into direct product of two-dimensional Poisson-Nijenhuis structures (ω_i, L_i).

Step 5. It remains to verify the statement in dimension 2 (Exercise).

More general result

Consider 4 elementary examples of compatible pairs:

Type 1. L has one real non-constant eigenvalue of multiplicity 2:

$$\omega = \mathrm{d}\, p \wedge \mathrm{d}\, x, \quad L = \lambda(x, p) \cdot \mathrm{Id}.$$

Type 2. L has one real constant eigenvalue of multiplicity $2k$;

$$\omega = \sum_{j=1}^{k} \mathrm{d}\, p_j \wedge \mathrm{d}\, x_j, \quad L = \lambda \cdot \mathrm{Id},\ \lambda \in \mathbb{R}.$$

Type 3. L has one pair of non-constant complex conjugate eigenvalues of multiplicity 2:

$$\omega = \mathrm{Re}\,(\mathrm{d}\, z \wedge \mathrm{d}\, w), \quad L = \alpha(z, w) \cdot \mathrm{Id} + \beta(z, w) \cdot J,$$

where $z = x + \mathrm{i}y, w = u + \mathrm{i}v$, J denotes the corresponding complex structure and $\alpha(z, w) + \mathrm{i}\beta(z, w)$ is a holomorphic function in z, w.

Type 4. L has one pair of constant complex conjugate eigenvalues of multiplicity $2k$:

$$\omega = \mathrm{Re}\,\left(\sum_{j=1}^{k} \mathrm{d}\, z_j \wedge \mathrm{d}\, w_j\right), \quad L = \alpha \cdot \mathrm{Id} + \beta \cdot J,$$

where $z_j = x_j + \mathrm{i}y_j, w_j = u_j + \mathrm{i}v_j$, J denotes the corresponding complex structure and $\alpha + \mathrm{i}\beta \in \mathbb{C}$, $\beta \neq 0$.

More general result

Theorem 8

Let ω and L be compatible. Suppose that L is semisimple and algebraically generic in a neighbourhood of $\mathrm{p} \in M$. Then the pair (ω, L) locally splits into a direct product of 'elementary blocks' of 4 types described above. If $\mathrm{d}\,\lambda(\mathrm{p}) \neq 0$ or $\mathrm{d}\,(\alpha(\mathrm{p}) + i\beta(\mathrm{p})) \neq 0$ for some real or complex eigenvalue, then in the corresponding 'elementary block' we may set $\lambda = x$ (see Type 1) and $\alpha + i\beta = z$ (see Type 3).

Important observation. To admit a compatible symplectic partner ω, a semisimple algebraically generic Nijenhuis operator L should satisfy the following additional condition: non-constant eigenvalues of L must be all of multiplicity two.

If L is algebraically generic but not necessarily semisimple, the description (rather non-trivial) of compatible pairs (ω, L) was obtained by F. Turiel under additional assumptions on the differentials of $\operatorname{tr} L^k$, $k = 1, \dots, n$. These assumptions basically mean that each eigenvalue is either constant or its differential does not vanish.

What about singular points?

If we want to study singularities of Poisson-Nijenhuis structures, then it is natural to ask which Nijenhuis operators admit compatible symplectic structures and what is a simultaneous canonical form for ω and L near a (possibly singular) point $\mathrm{p} \in M$?

What kind of singularities do we mean?

For a symplectic structure ω, all points $\mathrm{p} \in M$ are obviously equivalent to each other (in other words, there are no singular points). However it is not the case for an operator L since its algebraic properties may vary from point to point. Recall that $\mathrm{p} \in M$ is singular for L if the algebraic type of L changes at this point.

Among singular points, we distinguish a subclass of *differentially non-degenerate* singular points $\mathrm{p} \in M$, i.e., such that the differentials of the coefficients σ_k of the characteristic polynomial $\chi_L(t) = \det(t\,\mathrm{Id} - L)$ are linearly independent at p. Here

$$\chi_L(t) = \det(t \cdot \mathrm{Id} - L) = t^m + \sigma_1 t^{m-1} + \cdots + \sigma_{m-1} t + \sigma_m.$$

Differential non-degeneracy in this context

Relation (I'), however, implies that the eigenvalues of L have always even multiplicity, which in turn implies that L cannot be differentially non-degenerate. Recall that the characteristic polynomial is a full square

$$\chi_L(t) = (t^n + h_1 t^{n-1} + \cdots + h_{n-1} t + h_n)^2, \qquad m = 2n = \dim M,$$

and has therefore at most $\frac{1}{2} \dim M$ functionally independent coefficients. A natural analog of differential non-degeneracy in this setup is as follows:

The differentials $\mathrm{d}\, h_1(\mathrm{p}), \ldots, \mathrm{d}\, h_n(\mathrm{p})$ of the functions h_i are linearly independent at a given (singular) point $\mathrm{p} \in M$.

Question. Do such singularities appear in Poisson-Nijenhuis structures? If yes, can we describe local canonical forms for ω and L at such points?

Theorem 9

Let ω and L be compatible (i.e., define a Poisson-Nijenhuis structure on M^{2n}) and real analytic. Suppose that at a point $\mathrm{p} \in M^{2n}$, the differentials $\mathrm{d}\, h_1(\mathrm{p}), \dots, \mathrm{d}\, h_n(\mathrm{p})$ are linearly independent. Then there exists a local coordinate system $x_1, \dots, x_n, p_1, \dots, p_n$ such that $\omega = \sum_i \mathrm{d}\, x_i \wedge \mathrm{d}\, p_i$ and L is given by the matrix

$$\begin{pmatrix} A & 0_n \\ S & A^\top \end{pmatrix}, \tag{2}$$

where

$$A = \begin{pmatrix} -x_1 & 1 & 0 & \cdots & 0 \\ -x_2 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ -x_{n-1} & \vdots & & \ddots & 1 \\ -x_n & 0 & \cdots & \cdots & 0 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & -p_2 & -p_3 & \cdots & -p_n \\ p_2 & 0 & 0 & \cdots & 0 \\ p_3 & 0 & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ p_n & 0 & \cdots & \cdots & 0 \end{pmatrix}, \tag{3}$$

0_n is the zero $n \times n$-matrix, and $A^\top$ denotes the transposed of A.

Main fact to be used in the proof

Proposition 10

Let A be a (real-analytic) Nijenhuis operator which is gl-regular at a point $\mathsf{p} \in Q$. Assume that Ω is a closed (real-analytic) 2-form such that the form

$$\Omega_A(\xi, \eta) = \Omega(A\xi, \eta) + \Omega(\xi, A\eta) \tag{4}$$

is also closed. Then locally in some neighbourhood of $\mathsf{p} \in Q$ there exists a (real-analytic) function U such that

$$\Omega = \mathrm{d}\,(A^* \mathrm{d}\, U). \tag{5}$$

Remark. The gl-regularity condition is essential. It can be easily checked that the statement of Proposition 10 fails for the operator $A = \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix}$ at those points $\mathsf{p} \in Q$ where the eigenvalues collide, i.e., $x_1 = x_2$.

See proof in Lecture 19

Proposition 10 can be interpreted in terms of the Nijenhuis differential $\mathcal{L}_A : \Omega^k \to \Omega^{k+1}$ discussed at Lecture 19.
Recall that $\mathcal{L}_A$ is determined by the following properties

$$\begin{aligned}
&1.\ \mathcal{L}_A f = A^* \mathrm{d}\, f, \quad f \in \Omega^0,\\
&2.\ \mathcal{L}_A \text{ is linear},\\
&3.\ \mathcal{L}_A(\alpha \wedge \beta) = \mathcal{L}_A \alpha \wedge \beta + (-1)^k \alpha \wedge \mathcal{L}_A \beta, \quad \alpha \in \Omega^k, \beta \in \Omega^m,\\
&4.\ \mathcal{L}_A \mathrm{d} + \mathrm{d}\, \mathcal{L}_A = 0.
\end{aligned} \tag{6}$$

The explicit formula for the operator $\mathcal{L}_A : \Omega^k \to \Omega^{k+1}$ is as follows:

$$\mathcal{L}_A = [\mathrm{d}\,, i_A].$$

Since for 2-forms, i_A is given by (4), i.e. $i_A \Omega = \Omega_A$, we can reformulate Proposition 10 as follows:

Given 2-form Ω such that $\mathrm{d}\,\Omega = 0$ and $\mathcal{L}_A \Omega = 0$, one can always find a function U such that $\Omega = \mathrm{d}\, \mathcal{L}_A U = -\mathcal{L}_A \,\mathrm{d}\, U$.

That is exactly the statement of the main theorem from Lecture 19.

Proof of Theorem 9

Step 1. The characteristic polynomial of L is a full square:

$$\chi_L(t) = (t^n + h_1 t^{n-1} + \ldots + h_n)^2. \tag{7}$$

and the differentials of $h_1, \ldots, h_n$ are linearly independent. So we can take them as the first n coordinates $x_1, \ldots, x_n$ of a local coordinate system. We also know that these functions commute with respect to the Poisson bracket related to ω which, by Darboux theorem, implies local existence of functions $p_1, \ldots, p_n$ such that $(x_1, \ldots, x_n, p_1, \ldots, p_n)$ is a canonical coordinate system for ω.
Now the point is that we know how to "reconstruct" L from the coefficients of its characteristic polynomial (see formula (8) from Lecture 2). It is now straightforward (still some efforts are required!) to check that L takes the form (here we use both our "reconstruction formula" and algebraic conditions (I'))

$$L = \begin{pmatrix} A & 0 \\ \widehat{S} & A^\top \end{pmatrix} \tag{8}$$

with A as in (3) and $\widehat{S}$ being just a skew-symmetric matrix whose components may *a priori* depend on all variables.

Step 2. Next we use the differential compatibility condition (II') saying that the 2-form $\widetilde{\omega}(\cdot,\cdot) = \omega(L\cdot,\cdot)$ is closed. This form is given by

$$\widetilde{\omega} = \sum_{i=1}^{n} x_i \,\mathrm{d}\, x_1 \wedge \mathrm{d}\, p_i + \sum_{i,j=1}^{n} \widehat{S}_{ij} \,\mathrm{d}\, x_i \wedge \mathrm{d}\, x_j \qquad (9)$$

and its differential is

$$\mathrm{d}\,\widetilde{\omega} = -\sum_{i=2}^{n} \mathrm{d}\, x_1 \wedge \mathrm{d}\, x_i \wedge \mathrm{d}\, p_i + \sum_{i,j,k=1}^{n} \frac{\partial \widehat{S}_{ij}}{\partial x_k} \,\mathrm{d}\, x_k \wedge \mathrm{d}\, x_i \wedge \mathrm{d}\, x_j + \sum_{i,j,k=1}^{n} \frac{\partial \widehat{S}_{ij}}{\partial p_k} \,\mathrm{d}\, p_k \wedge \mathrm{d}\, x_i \wedge \mathrm{d}\, x_j . \qquad (10)$$

Substituting $\widehat{S} = S + T$, where S is as in (3), we observe that the first sum cancels and we obtain

$$\mathrm{d}\,\widetilde{\omega} = \sum_{i,j,k=1}^{n} \frac{\partial T_{ij}}{\partial p_k} \,\mathrm{d}\, p_k \wedge \mathrm{d}\, x_i \wedge \mathrm{d}\, x_j \; + \sum_{i,j,k=1}^{n} \frac{\partial T_{ij}}{\partial x_k} \,\mathrm{d}\, x_k \wedge \mathrm{d}\, x_i \wedge \mathrm{d}\, x_j . \quad (11)$$

Since $\widetilde{\omega}$ is closed, we see that T_{ij} does not depend on the p-variables, so that the form $\sum_{i,j=1}^{n} T_{ij}(x) dx_i \wedge dx_j$ can be viewed as a closed 2-form T on the local n-dimensional coordinate chart $(x_1, \ldots, x_n)$.

Step 3. To complete the proof, we will show that the form T can be 'killed' by a suitable canonical transformation of the form

$$(X_1, \dots, X_n, P_1, \dots, P_n) = \left(x_1, \dots, x_n,\; p_1 + \tfrac{\partial U}{\partial x_1}, \dots, p_n + \tfrac{\partial U}{\partial x_n}\right), \quad (12)$$

where U is a function of $x_1, \dots, x_n$. This transformation preserves ω so that we only need to look after the change of L.

The Jacobi matrix of coordinate transformation (12) is $J = \begin{pmatrix} \mathrm{Id} & 0 \\ \mathrm{d}^2 U & \mathrm{Id} \end{pmatrix}$, where $\mathrm{d}^2 U = \left(\frac{\partial^2 U}{\partial x_i \partial x_j}\right)$ is the Hessian matrix of U. Hence, after transformation (12) the matrix of L given by (8) takes the form

$$L_{\mathrm{new}} = JLJ^{-1} = \begin{pmatrix} \mathrm{Id} & 0 \\ \mathrm{d}^2 U & \mathrm{Id} \end{pmatrix} \begin{pmatrix} A & 0 \\ S + T & A^\top \end{pmatrix} \begin{pmatrix} \mathrm{Id} & 0 \\ -\mathrm{d}^2 U & \mathrm{Id} \end{pmatrix} = \begin{pmatrix} A & 0 \\ \widetilde{S} & A^\top \end{pmatrix},$$

where $\widetilde{S} = S + T + \mathrm{d}^2 U \cdot A - A^\top \cdot \mathrm{d}^2 U$. Taking into account the transformation of the p-coordinates in the S-block, we see that L_{new} takes the required form (2,3) if and only if the generating function U satisfies the following equation:

$$0 = T + \mathrm{d}^2 U \cdot A - A^\top \cdot \mathrm{d}^2 U + \mathrm{d}\, U \wedge \mathrm{d}\, x_1. \quad (13)$$

So we need to resolve it to find U.

Step 4. The latter equation can be rewritten in a more invariant form:

$$\mathrm{d}\,(A^{*}\mathrm{d}\,U) = T \tag{14}$$

that coincides with the PDE system (5) treated in Proposition 10.
In order to check the existence of a generating function U solving (14), it remains to verify two conditions from Proposition 10 imposed on T.

They both follow from the fact that $L = \begin{pmatrix} A & 0 \\ \widehat{S} & A^{\top} \end{pmatrix}$ with $\widehat{S} = S + T$ and A and S as in (3), is a Nijenhuis operator: the Nijenhuis relations for L are equivalent to the two geometric conditions we need, namely, that the 2-forms

$$T = \sum_{i,j=1}^{n} T_{ij}\mathrm{d}\,x_i \wedge \mathrm{d}\,x_j \ \text{ and } \ T_A = \sum_{i,j,k=1}^{n} \left(A^k{}_i T_{kj} + A^k{}_j T_{ik}\right)\mathrm{d}\,x_i \wedge \mathrm{d}\,x_j \tag{15}$$

are closed.

Hence, applying Proposition 10 guarantees the existence of U solving (14) and completes the proof of Theorem 9.

Exercises

Exercise 1

Remark after Proposition 10: show that for $A = \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix}$ fails at those points where $x_1 = x_2$. (Hint: the left hand side of (5) is not necessarily zero at such point, whereas the right hand side is).

Exercise 2

Let ω be a non-degenerate skew-symmetric form and L an operator such that the form $\widetilde{\omega}(\cdot, \cdot) = \omega(L\cdot, \cdot)$ is skew-symmetric. Prove that the characteristic polynomial of L is a full square (equivalently, the multiplicity of each eigenvalue of L is even).

Exercise 3

Prove Theorem 7 in dimension two (i.e., justify Step 5 in the proof of this theorem).

Exercise 4

Prove Theorem 8.