Nijenhuis Geometry

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Lecture 21:
Open Problems

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References

6. arXiv:1903.04603 [pdf, ps, other] math.DG

Nijenhuis Geometry

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Abstract: This work is the first, and main, of the series of papers in progress dedicated to Nienhuis operators, i.e., fields of endomorphisms with vanishing Nijenhuis tensor. It serves as an introduction to Nijenhuis Geometry that should be understood in much wider context than before: from local description at generic points to singularities and global analysis. The goal of the present paper is to introduce terminology, develop new important techniques (e.g., analytic functions of Nijenhuis operators, splitting theorem and linearisation), summarise and generalise basic facts (some of which are already known but we give new self-contained proofs), and more importantly, to demonstrate that the research programme proposed in the paper is realistic by proving a series of new, not at all obvious, results. △ Less

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7. arXiv:1804.03737 [pdf, ps, other] math.DS math.DG math.SG doi 10.1098/rsta.2017.0430

Open Problems, Questions, and Challenges in Finite-Dimensional Integrable Systems

Authors: A. Bolsinov, V. Matveev, E. Miranda, S. Tabachnikov

Abstract: The paper surveys open problems and questions related to different aspects of integrable systems with finitely many degrees of freedom. Many of the open problems were suggested by the participants of the conference "Finite-dimensional Integrable Systems, FDIS 2017" held at CRM, Barcelona in July 2017.

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[1] Bolsinov A., Matveev V., Miranda E., Tabachnikov S. *Open problems, questions and challenges in finite-dimensional integrable systems*. Phil. Trans. Royal Soc. A, 376 (2018) 20170430 http://dx.doi.org/10.1098/rsta.2017.0430

Three types of open problems

- Fundamental problems (that motivated the emergence of Nijenhuis Geometry)
  1. Local normal forms for Nijenhuis operators
  2. Singularities of Nijenhuis operators: their types, stability, linearisation problem, deformetions, etc.
  3. Global problems related to the behaviour (and existence!) of Nijenhuis operators on a manifold as a whole
  4. Theory of left symmetric algebras
  5. Applications of Nijenhuis Geometry
- Specific difficult questions
- Tasks/Problems for students and PhD students

In fact, at the moment it is hard to say how difficult these or those questions are... However, we clearly understand which of them are the most natural and important.

Normal forms: nilpotent case

Reminder.

- If a Nijenhuis operator L is similar to a nilpotent Jordan block at each point, then L can be brought to a constant form.
- Not every nilpotent Nijenhuis operator can be brought to a constant form (see Kobayashi's example in Lecture 6).
- From the algebraic viewpoint every nilpotent operator can be characterised by two natural flags:

$$\operatorname{Ker} L \subset \operatorname{Ker} L^2 \subset \dots \subset \operatorname{Ker} L^{s-1} \subset \operatorname{Ker} L^s = TM,$$

$$TM \supset \operatorname{Im} L \supset \operatorname{Im} L^2 \supset \dots \supset \operatorname{Im} L^{s-1} \supset \operatorname{Im} L^s = \{0\}.$$

- For Nijenhuis nilpotent operators, the distributions $\operatorname{Im} L^k$ are integrable, however, $\operatorname{Ker} L^k$ in general are not.
- Thompson theorem. Nijenhuis nilpotent operator can be brought to a constant form iff all the distributions $\operatorname{Ker} L^k$ are integrable.
- If $L^2 = 0$ and $\operatorname{rank} L = k \leq n/2$, then L can be brought to the form $L = \begin{pmatrix} 0_k & N(y) \\ 0 & 0_m \end{pmatrix}$. Classification of such operator is equivalent to classifications of k-dimensional distributions in m-dimensional space.

Normal forms: nilpotent case

Problem 1

Local classification/description of nilpotent Nijenhuis operators (of a fixed algebraic type).

Comments.

- If L is similar to a single Jordan 0-block, then by Thompson theorem (see Lecture 6) L locally reduces to a constant form.
- Additional question: do there exist Nijenhuis operator of this type that cannot be reduced to constant form globally? It is natural to assume that the topology of M is not trivial. For instance, M is a torus T^n or direct product $D^{n-1} \times S^1$.
- The problem is solved for operators L satisfying the conditions $L^2 = 0$ and $L^3 = 0$. The general case remains open. In fact, the problem reduces to a description of distributions with some additional properties ("wild problem"). But this "reduction" itself would be a very essential new result.
- An interesting fact: if the Jordan normal form of L consists of several blocks of the same size, then the problem becomes trivial: the operator reduces to constant form (Corollary from Thompson theorem, Lecture 6)

Normal forms: gl-regular operators

Reminder.

- An operator L is called gl-regular if one of the following holds
  - the gl-orbit $\mathcal{O}_L = \{PLP^{-1} \mid P \in GL(n, \mathbb{R})\}$ has maximal dimension
  - there is ξ such that $\xi, L\xi, \dots, L^{n-1}\xi$ are linearly independent
  - L can be brought to a companion form
- If L is a gl-regular Nijenhuis operator, then L can locally be brought to a companion form, i.e., in a neighborhood of any point (see Lectures 12 and 13). Such a companion form is uniquely determined by initial conditions: n functions of one variable, i.e., the values of the coefficients of the coefficients of the characteristic polynomial on the initial line $x(t) = (0, \dots, 0, t)$). The companion form L_{comp1} can be found from the initial conditions in purely algebraic way, i.e. without integration (see Lectures 13 and 14).
- Two different (Nijenhuis) companion forms L_{comp1} and L'_{comp1} can be equivalent. How to see this from initial conditions?
- Normal form for gl-regular Nijenhuis operators are known in dimension 2. All of them are polynomial (7 series), see Lecture 15.

Normal forms: gl-regular operators

Problem 2

Local classification of gl-regular Nijenhuis operators.

Comments.

- If follows from the Splitting Theorem that we may assume, without loss of generality, that at a given point p_0 our operator L is a nilpotent Jordan block.
- In dimension 2, the answer is known (see Lecture 15).
- Answer is known if at a given point p_0, the coefficients of the characteristic polynomial $\chi_L(t)$ are independent (i.e., if p_0 is differentially non-degenerate).
- If the coefficients $\sigma_1, \dots, \sigma_n$ of the characteristic polynomial are dependent, i.e. p_0 is a singular point for the characteristic map $S = (\sigma_1, \dots, \sigma_n) : M^n \to \mathbb{R}^n$, then the problem (almost) reduces to the classification of singularities for this map. These singularities are very special: the matrix $\left(\frac{\partial \sigma}{\partial x}\right)^{-1} L_{\mathrm{comp}}(\sigma) \left(\frac{\partial \sigma}{\partial x}\right)$ has a (smooth) limit as $x \to p_0$ and this limit is equal to $L(p_0)$.
- The problem is equivalent to a description of orbits of a certain algebroid.

Normal forms: Nijenhuis operators with linear components

Consider Nijenhuis operators $L(x)$ on $\mathbb{R}^n$ whose components are linear functions in Cartesian coordinates:

$$L = \left(L^i_j(x)\right), \qquad L^i_j(x) = \sum a^i_{jk} x^k.$$

Problem 3

Classify such Nijenhuis operators (for small n) and describe their normal forms (w.r.t. linear coordinate changes).

Equivalently: classify left symmetric algebras of small dimension.

Comments.

- In dimension 2, the problem has been solved by D. Burde and A. Konyaev (see Lecture 9).
- Some special sub-classes of left symmetric algebras (for example, Novikov algebras) are classified in dimension 3 and 4.
- In some sense, this classification is expected to resemble the classification of Lie algebras or associative algebras. But there number of LSA's is much larger as compared to the case Lie algebras or associative algebras).
- Computer algebra methods to be used.
- Even new examples would be very interesting to see and analyse.

Reminder.

- Recall that singular points of a Nijenhuis operator L are those where the algebraic type of L changes.
- For every operator $L(x)$ on a manifold M, we can define the characteristic map

$$\Phi = (\sigma_1, \dots, \sigma_n) : \mathrm{M} \to \mathbb{R}^n,$$

where $\sigma_1, \dots, \sigma_n$ are the coefficients of the characteristic polynomial

$$\chi_L(x) = \det(t \cdot \mathrm{Id} - L(x)) = t^n - \sigma_1(x)t^{n-1} - \dots - \sigma_{n-1}(x)t - \sigma_n(x).$$

- Assume that the coefficients $\sigma_1, \dots, \sigma_n$ of the characteristic polynomial of L are independent almost everywhere. Then L is defined by the formula (see formula (8) in Lecture 2):

$$L(x) = \left(\frac{\partial \sigma}{\partial x}\right)^{-1} L_{\mathrm{comp}}(\sigma)\left(\frac{\partial \sigma}{\partial x}\right), \quad L_{\mathrm{comp}}(\sigma) = \begin{pmatrix} \sigma_1 & 1 & & \\ \vdots & & \ddots & \\ \sigma_{n-1} & & & 1 \\ \sigma_n & 0 & \dots & 0 \end{pmatrix} \quad (2)$$

Problem 4 (Problem 5.14 in [1])

Describe (construct examples, study) singular points of a map $\Phi = (\sigma_1, \dots, \sigma_n) : \mathbb{R}^n \to \mathbb{R}^n$ satisfying the following "cancellation of denominator" property: all the elements of the matrix L are smooth functions including those points at which $\det\left(\frac{\partial \sigma}{\partial x}\right) = 0$.

For simplicity we may assume that $\sigma_1, \dots, \sigma_n$ are polynomials.

Comments.

- As you see, in the statement of the problem we do not mention Nijenhuis operators so that this problem is actually related to Singularity Theory.
- In this problem, the map $\Phi = (\sigma_1, \dots, \sigma_n)$ should be treated "up to a suitable coordinate change".
- What can we say about stability of such singularities (in the class of admissible singularities)?
- In this problem, there are two important particular cases (see next slides).

Singularities: generalisation of Example from Lecture 14

Perturbation of a Jordan block. Set

$$\lambda_{1,2} = \frac{2x_3}{(1-x_2) + \sqrt{(1-x_2)^2 - 4x_1x_3}}, \quad \lambda_3 = \frac{4x_3}{(1-2x_2) + \sqrt{(1-2x_2)^2 - 16x_1x_3}},$$

Then

$$L_{\text{comp1}} = \begin{pmatrix} f_1(x) & 1 & 0 \\ f_2(x) & 0 & 1 \\ f_3(x) & 0 & 0 \end{pmatrix} \quad \text{with} \quad \begin{aligned} f_1 &= \lambda_1 + \lambda_2 + \lambda_3, \\ f_2 &= -\lambda_1\lambda_2 - \lambda_2\lambda_3 - \lambda_3\lambda_1, \\ f_3 &= \lambda_1\lambda_2\lambda_3, \end{aligned}$$

is a Nijenhuis perturbation of the nilpotent 3×3 Jordan block J_0 under which J_0 splits into two Jordan blocks, of size 2 and 1 with non-constant eigenvalues.

Problem 5

Generalise this example to the case or arbitrary dimension. More precisely, construct an explicit perturbation $L(x) = J_0 + \dots$ of a Jordan block J_0 such that at a generic point $L(x)$ has a prescribed algebraic type, e.g., has s different eigenvalues with multiplicities $k_1, \dots, k_s$. Can one construct such an $L(x)$ which is polynomial in local coordinates?

If we assume, in addition, that L is gl-regular and coincides with a nilpotent Jordan block $J_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ at the singular point, then description of σ_1 and σ_2 was obtained in Lecture 15.
Namely, we have shown that there exists a coordinate system x, y such that

- $\sigma_2(x, y) = -\det L = \pm y^m$,
- $\sigma_1(x, y) = \operatorname{tr} L = \alpha x y^s + p_s(y)$.

Singularities: in dimension 2

In the previous Problem 4, consider the two-dimensional case: the map Φ is given by two functions $\sigma_1 = \operatorname{tr} L$, $\sigma_2 = -\det L$. For the sake of simplicity assume that at the singular point the differential $\mathrm{d}\Phi$ has rank one and moreover $\mathrm{d} \operatorname{tr} L \neq 0$ so that we can choose such coordinates (x, y) that $\sigma_1 = x$ and $\sigma_2 = f(x, y)$. Such a pair of functions ((singularity) is admissible if the operator L given by formula (2) is smooth.
The three components L^1_1, L^1_2 и L^2_2 of the matrix L are smooth automatically, and the remaining component takes the form

$$L^2_1 = \frac{f_x(x - f_x) - f}{f_y}$$

Problem 6 (Problem 5.13 in [1])

Describe the functions $f(x, y)$ (in a neighbourhood of the origin $(0, 0) \in \mathbb{R}^2$) such that $f_y(0, 0) = 0$ but $\frac{f_x(x-f_x)-f}{f_y}$ is a smooth function (can be extended up to a smooth function).

Singularities: special class of left symmetric algebras

In Problem 4, consider a special case when the coefficients $\sigma_1, \dots, \sigma_n$ of the characteristic polynomial are algebraically independent homogeneous polynomials in $x_1, \dots, x_n$ and, moreover, $\deg \sigma_k = k$. Then the description problem for admissible collections of functions (admissible singularities) becomes purely algebraic. It is easy to check that all elements of th ematrix $L(x) = \left(\frac{\partial \sigma}{\partial x}\right)^{-1} L_{\text{comp}}(\sigma)\left(\frac{\partial \sigma}{\partial x}\right)$ will be rational functions of the form $L^i_j = \dfrac{P_{ij}(x)}{D(x)}$, where $D = \det\left(\frac{\partial \sigma}{\partial x}\right)$, and P_{ij} is a polynomial of degree $\deg D + 1$.

Problem 7 (Problem 5.16 in [1])

Describe the collections of polynomials $\sigma_1, \dots, \sigma_n$ for which the denominators in all the fractions $\dfrac{P_{ij}(x)}{D(x)}$ cancel out, i.e., the elements of $L(x)$ turn out to be linear in $x_1, \dots, x_n$.

Comments

- An example is a collection of basis symmetric polynomials, i.e., coefficients of the polynomial $p(t) = \prod_i (t - x_i)$.
- Another example: $x_1, x_2x_n, x_3x_n^2, \ldots, x_n^n$.
- If $L_1(x)$ and $L_2(y)$ are examples then so is $L_1(x) \oplus L_2(y)$.
- Such operators $L(x)$ form a natural and apparently important class of left symmetric algebras.
- Another interesting question: are these left symmetric algebras non-degenerate?

Consider the family (pencil) of Nijenhuis operators of the form (some motivation will be given later)

$$L = \Big(a_{ij} + b_i x_j + x_i b_j + a x_i x_j\Big), \qquad a_{ij} = a_{ji}$$

or more generally

$$L = \Big(c^{ki}(a_{ij} + b_i x_j + x_i b_j + a x_i x_j)\Big), \quad c^{ki} = c^{ik}, a_{ij} = a_{ji}.$$

Problem 8

Describe all singular points of such operators (for all values of the parameters).

Comments.

- The problem has applications to the theory of Poisson brackets of hydrodynamic type.
- On the above collection of operators $L(x)$, we have a natural action of the group of affine transformations. It makes sense to first reduce an operator L to a canonical form. It is another important (and relatively easily solvable) problem.

Global aspects

Problem 9

Classify (real analytic) gl-regular Nijenhuis operator on two-dimensional closed surfaces.

Problem 10 (Problem 5.10 in [1])

Describe compact manifolds that admit *gl*-regular Nijenhuis operators.

Comments.

- In Lecture 15 there is a list of gl-regular operators on the torus and Klein bottle. Our conjecture is that this list essentially exhaust all the possibilities.
- Real analyticity is important! There exist smooth counterexamples to this conjecture.
- Problem 10 is much harder and requires serious preparatory work (see what we did in dim $= 2$).

Global aspects: more comments in dimension 2

In dimension 2, we have a reasonable conjecture about global structure of gl-regular operators:

- Operators of the form $\alpha \mathrm{Id} + \beta J$ ($\alpha, \beta \in \mathbb{R}$, $\beta \neq 0$) on orientable surfaces, where J is a complex structure.
- Non-trivial operators on the torus T^2 and Klein bottle K^2 from Examples 6,7 and 9 (Lecture 15).

To come to this conclusion/conjecture, we have done the following preparatory steps:

Step 1. local classification of gl-regular operators (explicit normal forms with polynomial entries)

Step 2. local description of the singular set

Step 3. important observation: not all normal forms are admissible on compact manifolds

Step 4. global properties of the singular

Step 5. examples

Step 6. conjecture

It is very likely that in other dimensions we will have to follow the same scheme.

In many problems there appear operators with singular points that do not satisfy the gl-regularity condition but still possess some good properties. One of them is linearisability.

Reminder.

Consider a singular point p_0 (of scalar type) and expand L into a Taylor series

$$L(p_0) = \lambda\,\mathrm{Id} + L_{lin}(x) + L_2(x) + \dots .$$

We say that this singularity is linearisable if all the higher order terms (except for $\lambda\,\mathrm{Id} + L_{lin}(x)$) can be killed by by an appropriate coordinate transformation.

Problem 11

Which two-dimensional surfaces admit Nijenhuis operators with linearisable singular points? The same question under additional condition that $L(x)$ is diagonalisable over $\mathbb{R}$ at each point x.

Comments.

- Of course, we mean non-trivial examples, different from scalar operators $f(x) \cdot \mathrm{Id}$ and complex structures (and maybe some others...).
- This problem can certainly be solved. At each singular point one can define an analog of an index and then use some classical techniques. Since these points are assumed to be linearisable, the corresponding indices can be computed for each of them (the list of all possible linearisations can be found in Lecture 9).
- Examples exist on the sphere S^2 and torus T^2 (they are related to integrable geodesic flows).
- If we omit linearisability condition, then $\mathbb{R}$-diagonalisable Nijenhuis operators should exists on all closed surfaces. However, it is not yet clear how to construct them explicitly (Problem 5.12 in [1])!

Global aspects

In many areas of Differential Geometry, the "partition of unity" concept turns out to be quite useful and efficient. The role of this concept in Nijenhuis Geometry is yet to be clarified...

Problem 12 (Problem 5.12 in [1])

Consider a Nijenhuis operator $L(x)$ of fixed algebraic type, given on a small neighbourhood $U \subset \mathbb{R}^n$. Is it possible to extend this operator $L(x)$ to the whole space $\mathbb{R}^n$ in such a way that it remains a Nijenhuis operator and it is identically zero outside a certain "larger" domain V (where $U \subset V$)?

Comments.

- The answer depend on the algebraic type: if $L(x)$ is diagonalisable over $\mathbb{R}$ and has n different roots, then the answer is positive. If there are complex roots, the answer is negative.
- For a Jordan block, the question is open.
- The question can be modified. For instance, a Nijenhuis operator is defined on an annulus and we need to extend it inside (to get a Nijenhuis operator on the disc).
- A positive answer would give us a possibility to construct Nijenhuis operators on manifolds with "complicated" topology.