Nijenhuis Geometry

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Lecture 22:
Open Problems (Part 2)

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Abstract: This work is the first, and main, of the series of papers in progress dedicated to Nienhuis operators, i.e., fields of endomorphisms with vanishing Nijenhuis tensor. It serves as an introduction to Nijenhuis Geometry that should be understood in much wider context than before: from local description at generic points to singularities and global analysis. The goal of the present paper is to introduce terminology, develop new important techniques (e.g., analytic functions of Nijenhuis operators, splitting theorem and linearisation), summarise and generalise basic facts (some of which are already known but we give new self-contained proofs), and more importantly, to demonstrate that the research programme proposed in the paper is realistic by proving a series of new, not at all obvious, results. △ Less

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7. arXiv:1804.03737 [pdf, ps, other] math.DS math.DG math.SG doi 10.1098/rsta.2017.0430

Open Problems, Questions, and Challenges in Finite-Dimensional Integrable Systems

Authors: A. Bolsinov, V. Matveev, E. Miranda, S. Tabachnikov

Abstract: The paper surveys open problems and questions related to different aspects of integrable systems with finitely many degrees of freedom. Many of the open problems were suggested by the participants of the conference "Finite-dimensional Integrable Systems, FDIS 2017" held at CRM, Barcelona in July 2017.

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[1] Bolsinov A., Matveev V., Miranda E., Tabachnikov S. *Open problems, questions and challenges in finite-dimensional integrable systems*. Phil. Trans. Royal Soc. A, 376 (2018) 20170430 http://dx.doi.org/10.1098/rsta.2017.0430

Compatible Poisson brackets/structures

Definition.

Two Poisson structures P_1 and P_2 are compatible if the sum $P_1 + P_2$ is a Poisson structure also.

A vector field $\mathcal{X}$ is called bi-Hamiltonian if it is Hamiltonian w.r.t. two compatible Poisson brackets P_1 and P_2:

$$\mathcal{X} = P_1 \,\mathrm{d}\, f_1 = P_2 \,\mathrm{d}\, f_2.$$

As a rule, bi-Hamiltonian systems are integrable and possess many other remarkable properties.

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Geodesically equivalent metrics

Definition. Two (pseudo)-Riemannian metrics g_1 and g_2 are projectively equivalent if they have the same geodesics (as unparametrised curves).

Dini Theorem: Two Riemannian metrics g_1 and g_2 in dimension 2 are geodesically equivalent if and only if in an appropriate coordinate system:

$$g_1 = (v(y) - u(x))(\mathrm{d}\,x^2 + \mathrm{d}\,y^2) \quad \text{and} \quad g_2 = \left(\frac{1}{u(x)} - \frac{1}{v(y)}\right)\left(\frac{\mathrm{d}\,x^2}{u(x)} + \frac{\mathrm{d}\,y^2}{v(y)}\right).$$

Levi-Civita Theorem: Riemannian, arbitrary dimension, generic point.

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Poisson structures of hydrodynamic type

The notion of a Hamiltonian system can be naturally introduced for systems of PDEs. In particular, a quasi-linear system of hydrodynamic type:

$$u_t = A(u)u_x, \quad \text{where } u = \begin{pmatrix} u^1(t,x) \\ \vdots \\ u^n(t,x) \end{pmatrix}, \text{ and } A(u) = \left(A^i_j(u)\right) \text{ is}$$

an operator on a manifold M with local coordinates u, will be Hamiltonian if

$$A^i_j = \nabla^i \nabla_j h = g^{ij} \frac{\partial^2 h}{\partial x^s \partial x^j} - \Gamma^{is}_j \frac{\partial h}{\partial x_s},$$

where ∇ is the Levi-Civita connection for a flat metric g and $h : \mathrm{M} \to \mathbb{R}$. In this sense, g defines a Poisson structure of hydrodynamic type.

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Nijenhuis operators as partners of other geometric structures

- Consider two compatible Poisson structures P_1 and P_2, one of which is non-degenerate. Then the operator $L = P_2 P_1^{-1}$ is Nijenhuis.
- Consider two geodesically equivalent metrics g_1 and g_2. Then the operator $L = \left(\frac{\det g_2}{\det g_1}\right)^{\frac{1}{n+1}} g_2^{-1} g_1$ is Nijenhuis.
- Consider two compatible Poisson brackets of hydrodynamic type, determined by two flat metrics g_1 and g_2. Then the operator $L = g_2^{-1} g_1$ is Nijenhuis.

For each of these situations it makes sense to introduce the following terminology:

- L is compatible with a Poisson structure P, if P and LP are compatible Poisson structures.
- L is geodesically compatible with g, if the metrics g and $\frac{1}{|\det L|} g L^{-1}$ are geodesically equivalent.
- L is Poisson compatible with a flat metric g, if gL^{-1} is flat and the metrics g and gL^{-1} define a pair of compatible Poisson brackets of hydrodynamic type.

Which Nijenhuis L can find a suitable partner?

Problem 13

Let L be a Nijenhuis operator. Does it admit

- compatible Poisson structure P,
- geodesically compatible metric g,
- Poisson compatible (flat) metric g ?

This question makes sense both in global and local context.

Comments.

- If the algebraic type of an operator L is locally constant (i.e. we consider a non-singular point), then the answer is essentially known (Turiel and AB and V. Matveev) in the first two cases. It the third case the answer does not seem to be clear...
- Thus the question is basically related to singular points: which singularities of the operator L are admissible in this context?
- If L is differentially non-degenerate at a given singular point, then the partner for L exists in each of the above cases ([Nij1], [ApplNij1], [ApplNij2]).
- In the case of geodesic compatibility, all admissible singularities are know if g is positive definite (V. Matveev).

Pencils of Nijenhuis operators

Definition. Two Nijenhuis operators L_1 and L_2 are called compatible, if their sum is again a Nijenhuis operator.
A pencil of Nijenhuis operators $\mathcal{L}$ is a vector subspace of the space $\mathrm{End}(TM)$ of all operators, which consist of Nijenhuis operators.

A natural general question is description of examples and classification of Nijenhuis pencils (at generic points). Maximal pencils are of particular interest (i.e. those which cannot be extended). Two examples:

- diagonal pencil $\left\{L = \mathrm{diag}\big(f_1(x_1), f_2(x_2), \dots, f_n(x_n)\big)\right\}$
- pencil of operators of th eform $\left\{L = \left(a_{ij} + b_i x_j + x_i b_j + a x_i x_j\right), a_{ij} = a_{ji}\right\}$ or, more generally, $\left\{L = \left(c^{ki}(a_{ij} + b_i x_j + x_i b_j + a x_i x_j)\right), a_{ij} = a_{ji}\right\}$

Since we do not know almost anything about Nijenhuis pencils, a natural task, at the moment, would be just funding and studying examples.

Problem 14

- Construct examples of (maximal) pencils that contain the diagonal operator $\{L = \mathrm{diag}(x_1, \ldots, x_n)\}$ (see the two examples above)
- Which operators are compatible with the diagonal operator? With operators of the form $f(x) \cdot \mathrm{Id}$?
- What is the minimal dimension for a maximal pencil? Do there exist three-dimensional maximal pencils?
- Do there exist pencils $\mathcal{L}$ such that each $L \in \mathcal{L}$ can be reduced to a constant form, but all together cannot?
- Shift of argument (version 1): Let $L = \left(c^i_{jk} x^j\right)$ be a Nijenhuis operator that is linear in local coordinates. Then $L_a = \left(c^i_{jk} a^j\right)$ is compatible with L for any $a \in \mathbb{R}^n$, and all together they form a pencil. What can we say about its extension up to a maximal pencil?
- Shift of argument (version 2): Let $\mathfrak{g}$ be a Frobenius Lie algebra, i.e. the natrix $\mathcal{A}_x = \left(c^k_{ij} x^k\right)$ is non-degenerate at a generic point. The Nijenhuis operators of the form $\mathcal{A}_a \mathcal{A}_x^{-1}$, $a \in \mathbb{R}^n$ form a pencil of dimension n. What can we say about its maximality?

Nijenhuis cohomologies

For Nijenhuis manifolds, one can introduce another differentiation operator $\mathcal{L}_L = \mathrm{d}_L : \Omega^k(M) \to \Omega^{k+1}(M)$ on the complex of exterior differential forms, with two basic properties (see Lectures 18 and 19):

- $\mathcal{L}_L f = L^*(\mathrm{d}f) = L_i^\alpha \frac{\partial f}{\partial x^\alpha} \mathrm{d}x^i$;
- $\mathcal{L}_L \mathrm{d}x^i = -\mathrm{d}\big(L^* \mathrm{d}x^i\big) = -\mathrm{d}\big(L^i_\alpha \mathrm{d}x^\alpha\big)$.

If L is a Nijenhuis operator, then $\mathcal{L}_L^2 = 0$ and we can naturally define the cohomology groups $H^k_L(M)$.

Problem 15

What can we say about these cohomologies in local setting when M is a neighborhood of a singular point of the Nijenhuis operator L? The simplest case to start with is $L(x, y)$ with components linear in x and y in dimension 2. All these case are described in Lecture 9 (list of left symmetric algebras in 2D).

Problem 16

Compute the cohomologies for known global examples (complex structure, partners of geodesically equivalent metrics, operators on two-dimensional surfaces).

Problem 17

Verify the following conjecture: Let L be gl-regular and $\mathrm{d}_L\omega = \mathrm{d}\omega = 0$ for $\omega \in \Omega^k(M)$. Then there exists $\alpha \in \Omega^{k-2}$ such that $\omega = \mathrm{d}\mathrm{d}_L\alpha$.

What else?

- Nijenhuis Zoo: collecting various samples of Nijenhuis operators.
  - Examples of Nijenhuis operators with polynomial components (quadratic, cubic, ...)
  - Global examples on interesting manifolds
  - Examples of compatible Nijenhuis operators
  - etc.
- Many questions related to left symmetric algebras (separate huge topic): classification of LSA and studying their non-degeneracy.
- How are singularities of Nijenhuis operators related with the singularities of the Hamiltonian systems associated with them (Problem 5.16 in [1]). Can we check non-degeneracy of singularities by using Nijenhuis operators?
- Topology of manifolds admitting geodesically equivalent metrics.
- Studying systems of hydrodynamic type $u_t = A(u)u_x$ and (perhaps!) proving the conjecture [1] on geodesic flows on the torus T^2.

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