Nijenhuis Geometry

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Lecture 2:
Nijenhuis torsion and its basic properties

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Outline of Lecture 2

- Nijenhuis torsion: equivalent definitions
- Nijenhuis torsion in local coordinates
- Basic properties of Nijenhuis operators
- Properties of the characteristic polynomial
- Properties of the eigenvalues

Nijenhuis torsion: equivalent definitions

Let L be a $(1,1)$-tensor field (operator) on a smooth manifold M. The Nijenhuis torsion $\mathcal{N}_L$ of the operator L is a $(1,2)$-tensor that can be defined in several equivalent ways.

Definition

- As a vector-valued 2-form:

$$\mathcal{N}_L(\xi,\eta) = L^2[\xi,\eta] - L[L\xi,\eta] - L[\xi,L\eta] + [L\xi,L\eta].$$

- As a map from 'vector fields' to 'operators':

$$\mathcal{N}_L : \ \xi \mapsto \mathcal{L}_{L\xi} L - L\mathcal{L}_\xi L$$

- As a map from "1-forms" to "2-forms":

$$\mathcal{N}_L : \ \alpha \mapsto \beta, \quad \text{where}$$

$$\beta(\cdot\,,\cdot) = -\mathrm{d}\,({L^*}^2\alpha)(\cdot\,,\cdot) - \mathrm{d}\,\alpha(L\cdot\,,L\cdot) + \mathrm{d}\,(L^*\alpha)(L\cdot\,,\cdot) + \mathrm{d}\,(L^*\alpha)(\cdot\,,L\cdot).$$

- In local coordinates (summation over repeated indices s is assumed):

$$(\mathcal{N}_L)^i_{jk} = L^s_j \frac{\partial L^i_k}{\partial x^s} - L^s_k \frac{\partial L^i_j}{\partial x^s} - L^i_s \frac{\partial L^s_k}{\partial x^j} + L^i_s \frac{\partial L^s_j}{\partial x^k}.$$

Nijenhuis torsion as a map from 'vector fields' to 'operators'

Since $\mathcal{N}_L$ is a tensor of type $(1,2)$ we can treat it as a map

$$\text{vector field } \xi \mapsto \text{linear operator } A$$

defined by $\xi^i \mapsto A^k_j = \sum_i (\mathcal{N}_L)^k_{ij}\, \xi^i$ or in more invariant terms:

$$A\eta = \mathcal{N}_L(\xi, \eta). \tag{1}$$

We need to show that $A = \mathcal{L}_{L\xi} L - L\mathcal{L}_\xi L$, where $\mathcal{L}_\xi$ is the Lie derivative. Let us apply the operator $\mathcal{L}_{L\xi} L - L\mathcal{L}_\xi L$ to a vector field η to verify (1):

$$(\mathcal{L}_{L\xi} L - L\,\mathcal{L}_\xi L)\,\eta = (\mathcal{L}_{L\xi} L)\,\eta - L(\mathcal{L}_\xi L)\,\eta =$$

now using $\boxed{\mathcal{L}_u(Av) = (\mathcal{L}_u A)v + A(\mathcal{L}_u v)}$:

$$\big(\mathcal{L}_{L\xi}(L\,\eta) - L\mathcal{L}_{L\xi}\eta\big) - L\big(\mathcal{L}_\xi(L\eta) - L\mathcal{L}_\xi\eta\big) =$$

next using $\boxed{\mathcal{L}_u v = [u, v]}$:

$$[L\xi, L\eta] - L[L\xi, \eta] - L[\xi, L\eta] + L^2[\xi, \eta] = \mathcal{N}_L(\xi, \eta).$$

Nijenhuis torsion as a map from '1-forms' to '2-forms'

Since $\mathcal{N}_L$ is a tensor of type $(1,2)$ we can treat it as a map

$$\text{1-form } \alpha \mapsto \text{2-form } \beta$$

defined by $\alpha_k \;\mapsto\; \omega_{ij} = (\mathcal{N}_L)^k_{ij}\,\alpha_k$ or, in more invariant terms, $\omega(\xi,\eta) = \alpha\big(\mathcal{N}_L(\xi,\eta)\big)$. Thus, we need to show that

$$-\mathrm{d}\,(L^{*2}\alpha)(\xi,\eta) - \mathrm{d}\,\alpha(L\xi, L\eta) + \mathrm{d}\,(L^*\alpha)(L\xi,\eta) + \mathrm{d}\,(L^*\alpha)(\xi, L\eta) = \alpha\big(\mathcal{N}_L(\xi,\eta)\big)$$

We compute the l.h.s. using $\boxed{\mathrm{d}\,\alpha(u,v) = u(\alpha(v)) - v(\alpha(u)) - \alpha([u,v])}$:

$$-\xi\big(L^{*2}\alpha(\eta)\big) + \eta\big(L^{*2}\alpha(\xi)\big) + L^{*2}\alpha([\xi,\eta]) - L\xi\big(\alpha(L\eta)\big) + L\eta\big(\alpha(L\xi)\big) + \alpha([L\xi, L\eta]) +$$

$$L\xi\big(L^*\alpha(\eta)\big) - \eta\big(L^*\alpha(L\xi)\big) - L^*\alpha([L\xi,\eta]) + \xi\big(L^*\alpha(L\eta)\big) - L\eta\big(L^*\alpha(\xi)\big) - L^*\alpha([\xi, L\eta])$$

Nijenhuis torsion as a map from ‘1-forms’ to ‘2-forms’

Since $\mathcal{N}_L$ is a tensor of type $(1,2)$ we can treat it as a map

$$\text{1-form } \alpha \mapsto \text{2-form } \beta$$

defined by $\alpha_k \;\mapsto\; \omega_{ij} = (\mathcal{N}_L)^k_{ij}\,\alpha_k$ or, in more invariant terms, $\omega(\xi,\eta) = \alpha\big(\mathcal{N}_L(\xi,\eta)\big)$. Thus, we need to show that

$$-\mathrm{d}\,({L^*}^2\alpha)(\xi,\eta) - \mathrm{d}\,\alpha(L\xi, L\eta) + \mathrm{d}\,(L^*\alpha)(L\xi,\eta) + \mathrm{d}\,(L^*\alpha)(\xi, L\eta) = \alpha\big(\mathcal{N}_L(\xi,\eta)\big)$$

We compute the l.h.s. using $\boxed{\mathrm{d}\,\alpha(u,v) = u(\alpha(v)) - v(\alpha(u)) - \alpha([u,v])}$:

$$-\xi\big({L^*}^2\alpha(\eta)\big) + \eta\big({L^*}^2\alpha(\xi)\big) + {L^*}^2\alpha\big([\xi,\eta]\big) - L\xi\big(\alpha(L\eta)\big) + L\eta\big(\alpha(L\xi)\big) + \alpha\big([L\xi, L\eta]\big) +$$

$$L\xi\big(L^*\alpha(\eta)\big) - \eta\big(L^*\alpha(L\xi)\big) - L^*\alpha\big([L\xi,\eta]\big) + \xi\big(L^*\alpha(L\eta)\big) - L\eta\big(L^*\alpha(\xi)\big) - L^*\alpha\big([\xi, L\eta]\big)$$

and next using $\boxed{A^*\alpha(u) = \alpha(Au)}$:

$$\alpha\Big(L^2[\xi,\eta] + [L\xi, L\eta] - L[L\xi,\eta] - L[\xi, L\eta]\Big) = \alpha\Big(\mathcal{N}_L(\xi,\eta)\Big).$$

Nijenhuis torsion in local coordinates

Here we only need to show that the components $(\mathcal{N}_L)^k_{ij}$ of the Nijenhuis torsion in a local coordinate system $x^1, \dots, x^n$ take the following form

$$(\mathcal{N}_L)^i_{jk} = L^s_j \frac{\partial L^i_k}{\partial x^s} - L^s_k \frac{\partial L^i_j}{\partial x^s} - L^i_s \frac{\partial L^s_k}{\partial x^j} + L^i_s \frac{\partial L^s_j}{\partial x^k}.$$

Let ∂_{x^j} and ∂_{x^k} be basis vector fields. Since $\mathcal{N}_L(\partial_{x^j}, \partial_{x^k}) = \sum_i (\mathcal{N}_L)^i_{jk} \partial_{x^i}$, we only need to compute $\mathcal{N}_L(\partial_{x^j}, \partial_{x^k})$. Let us do it:

$$\mathcal{N}_L(\partial_{x^j}, \partial_{x^k}) = L^2[\partial_{x^j}, \partial_{x^k}] - L[L\partial_{x^j}, \partial_{x^k}] - L[\partial_{x^j}, L\partial_{x^k}] + [L\partial_{x^j}, L\partial_{x^k}] =$$

we use $\boxed{L\partial_{x^i} = L^k_i \partial_{x^k}}$ (with summation over k) and $\boxed{[\partial_{x^i}, f^k \partial_{x^k}] = \frac{\partial f^k}{\partial x^i} \partial_{x^k}}$

$$-L\left[L^s_j \partial_{x^s}, \partial_{x^k}\right] - L\left[\partial_{x^j}, L^s_k \partial_{x^s}\right] + \left[L^s_j \partial_{x^s}, L^i_k \partial_{x^i}\right] =$$

$$\frac{\partial L^s_j}{\partial_{x^k}} L\partial_{x^s} - \frac{\partial L^s_k}{\partial_{x^j}} L\partial_{x^s} + L^s_j \frac{\partial L^i_k}{\partial x^s} \partial_{x^i} - L^i_k \frac{\partial L^s_j}{\partial_{x^i}} \partial_{x^s} = \left(\frac{\partial L^s_j}{\partial_{x^k}} L^i_s - \frac{\partial L^s_k}{\partial_{x^j}} L^i_s + L^s_j \frac{\partial L^i_k}{\partial x^s} - L^s_k \frac{\partial L^i_j}{\partial_{x^s}} \right) \partial_{x^i}$$

as was to be proved (in the last term we interchanged i and s).

One useful remark

Since $(\mathcal{N}_L)^i_{jk} = L^s_j \frac{\partial L^i_k}{\partial x^s} - L^s_k \frac{\partial L^i_j}{\partial x^s} - L^i_s \frac{\partial L^s_k}{\partial x^j} + L^i_s \frac{\partial L^s_j}{\partial x^k}$ is a tensor with 3 indices, we may use it to produce "new" tensors by applying elementary tensor operations, e.g.,

$$L^i_s\,(\mathcal{N}_L)^s_{jk}, \quad L^s_j\,(\mathcal{N}_L)^i_{sk}, \quad (\mathcal{N}_L)^i_{js}(\mathcal{N}_L)^s_{kp}, \quad \dots$$

But the simplest option is the 1-form $(\mathcal{N}_L)^i_{ik}$. Let us see what the geometric meaning of this form is:

$$(\mathcal{N}_L)^i{}_{ik} = L^s_i \frac{\partial L^i{}_k}{\partial x^s} - L^s_k \frac{\partial L^i{}_i}{\partial x^s} - L^i{}_s \frac{\partial L^s_k}{\partial x^i} + L^i{}_s \frac{\partial L^s_i}{\partial x^k} =$$

$$L^s_i \frac{\partial L^i_k}{\partial x^s} - L^s_k \frac{\partial L^i_i}{\partial x^s} - L^i_s \frac{\partial L^s_k}{\partial x^i} + L^i_s \frac{\partial L^s_i}{\partial x^k} = -L^s_k \frac{\partial L^i_i}{\partial x^s} + L^i_s \frac{\partial L^s_i}{\partial x^k} =$$

$$-L^s_k \frac{\partial \,\mathrm{tr}\, L}{\partial x^s} + \frac{1}{2} \frac{\partial (L^i_s L^s_i)}{\partial x^k} = \left(-L^*(\mathrm{d}\ \mathrm{tr}\, L) + \frac{1}{2} \mathrm{d}\ \mathrm{tr}\, L^2 \right)_k$$

Conclusion. Contraction of the Nijenhuis torsion gives the 1-form $\frac{1}{2}\mathrm{d}\ \mathrm{tr}\, L^2 - L^*(\mathrm{d}\ \mathrm{tr}\, L)$. In particular, for Nijenhuis operators we have

$$\mathrm{d}\ \mathrm{tr}\, L^2 = 2L^*(\mathrm{d}\ \mathrm{tr}\, L).$$

Some preliminaries and reminders

- If L is Nijenhuis, then $c \cdot L$ and $L + c \cdot \mathrm{Id}$, $c \in \mathbb{R}$, are Nijenhuis also.
- $\frac{d}{dt}\left(A(t)B(t)\right) = \left(\frac{d}{dt}A(t)\right) B(t) + A(t)\frac{d}{dt}B(t)$.
- Similarly, $\mathcal{L}_\xi(AB) = (\mathcal{L}_\xi A)B + A\,\mathcal{L}_\xi B$.
- $\frac{d}{dt}\det A(t) = \operatorname{tr}\left(\widehat{A}(t)\frac{d}{dt}A(t)\right)$, where $\widehat{A}$ is the adjugate matrix of A.
- Similarly, $\mathcal{L}_\xi \det L = \operatorname{tr}\left(\widehat{A}\,\mathcal{L}_\xi A\right)$.
- $\langle A^*(\mathrm{d}\, f), \xi\rangle = \langle \mathrm{d}\, f, A\,\xi\rangle$ (to compute $A^*(\mathrm{d}\, f)$ in practice, one simply multiply the row-vector $\left(\frac{\partial f}{\partial x^1}, \dots, \frac{\partial f}{\partial x^n}\right)$ with the matrix A on the right).
- The coefficients of the characteristic polynomial $\chi_{A(x)}(t) = \det\big(t \cdot \mathrm{Id} - A(x)\big)$ are smooth functions in x.
- Let $p_x(t) = t^n - \sigma_1(x)t^{n-1} - \dots - \sigma_{n-1}(x)t - \sigma_n(x)$ be a polynomial with coefficients smoothly depending on x. Assume that for all $x \in U$ (small neighbourhood) the multiplicities of roots of $p_x(t)$ are constant. Then each root $\lambda_i(x)$ is a smooth function of x (including complex roots).

Basic facts

Let L be a Nijenhuis operator, then

- For any polynomial $p(\cdot)$ with constant coefficients, the operator $p(L)$ is also Nijenhuis. Moreover, the same holds true for any real analytic matrix function $f(L)$.
- $$\mathrm{d}\,(\mathrm{tr}\, L^k) = k(L^*)^{k-1}\mathrm{d}\ \mathrm{tr}\, L,$$
  where $L^* : T_q^*M \to T_q^*M$ denotes the operator dual to L.
- $$L^*\mathrm{d}\,(\det L) = \det L \cdot \mathrm{d}\ \mathrm{tr}\, L.$$
- More generally, the differential of the characteristic polynomial $\chi(t) = \det(t \cdot \mathrm{Id} - L)$ satisfies the following relation:
  $$L^*\big(\mathrm{d}\,\chi(t)\big) - t \cdot \mathrm{d}\,\chi(t) = \chi(t) \cdot \mathrm{d}\ \mathrm{tr}\, L.$$
- Let $\lambda(x)$ be a smooth eigenvalue of L. Then
  $$\big(L - \lambda(x) \cdot \mathrm{Id}\big)^*\mathrm{d}\,\lambda(x) = 0.$$

Property 1

Proposition

Let L be a Nijenhuis operator, then for any polynomial $p(\cdot)$ with constant coefficients, the operator $p(L)$ is also Nijenhuis. In other words, $\mathcal{N}_L = 0$ implies $\mathcal{N}_{p(L)} = 0$.

Proof.

We use the condition $\mathcal{N}_L = 0$ in the form $\mathcal{L}_{L\xi} L = L\mathcal{L}_\xi L$. This identity implies

$$\mathcal{L}_{L^n\xi} L = \mathcal{L}_{L(L^{n-1}\xi)} L = L\mathcal{L}_{L^{n-1}\xi} L = \cdots = L^n \mathcal{L}_\xi L \tag{2}$$

and, therefore, by linearity $\mathcal{L}_{p(L)\xi} L = p(L)\mathcal{L}_\xi L$. Thus, we have

$$(\mathcal{L}_{p(L)\xi} - p(L)\mathcal{L}_\xi) L = 0$$

Now consider the expression $\mathcal{D} = \mathcal{L}_{p(L)\xi} - p(L)\mathcal{L}_\xi$ as a "first order differential operator" which satisfies the obvious property $\mathcal{D}(L^n) = \mathcal{D}(L^{n-1})\, L + L^{n-1}\mathcal{D}(L)$. Hence $\mathcal{D}(L) = 0$ immediately implies $\mathcal{D}(p(L)) = 0$, i.e.,

$$\left(\mathcal{L}_{p(L)\xi} - p(L)\mathcal{L}_\xi\right) p(L) = 0,$$

which is exactly the desired condition $\mathcal{N}_{p(L)} = 0$. □

Property 2

Proposition

$$\mathrm{d}\,(\mathrm{tr}\,L^k) = k(L^*)^{k-1}\mathrm{d}\ \mathrm{tr}\,L, \tag{3}$$

and more generally,

$$\mathrm{d}\,\big(\mathrm{tr}\,p(L)\big) = p'(L)^*\mathrm{d}\ \mathrm{tr}\,L, \tag{4}$$

where $L^* : T_q^*M \to T_q^*M$ denotes the operator dual to L and $p'(\cdot)$ is the derivative of $p(\cdot)$.

Proof.

(3) can be rewritten as the following identity (for an arbitrary ξ)

$$\langle \mathrm{d}\ \mathrm{tr}\,L^k, \xi\rangle = \langle k(L^*)^{k-1}\mathrm{d}\ \mathrm{tr}\,L, \xi\rangle \quad \text{or} \quad \mathcal{L}_\xi(\mathrm{tr}\,L^k) = k\mathcal{L}_{L^{k-1}\xi}\,\mathrm{tr}\,L.$$

To prove it, we use (2) (with n replaced with $k-1$). We have

$$\mathcal{L}_\xi\,\mathrm{tr}\,L^k = \mathrm{tr}\,\mathcal{L}_\xi L^k = k\,\mathrm{tr}\,\big(L^{k-1}\mathcal{L}_\xi L\big) \overset{(2)}{=} k\,\mathrm{tr}\,\big(\mathcal{L}_{L^{k-1}\xi}L\big) = k\mathcal{L}_{L^{k-1}\xi}\,\mathrm{tr}\,L$$

which is equivalent to (3). □

Properties 3 and 4

Proposition

Let L be a Nijenhuis operator. Then for any vector field ξ we have

$$\mathcal{L}_{L\xi}(\det L) = \det L \cdot \mathcal{L}_{\xi} \operatorname{tr} L$$

or, equivalently,

$$L^* \mathrm{d}\,(\det L) = \det L \cdot \mathrm{d}\, \operatorname{tr} L. \tag{5}$$

More generally, the differential of the characteristic polynomial $\chi(t) = \det(t \cdot \mathrm{Id} - L)$ (viewed as a smooth function on M with t as a formal parameter) satisfies the following relation:

$$L^* \big(\mathrm{d}\,\chi(t)\big) - t \cdot \mathrm{d}\,\chi(t) = \chi(t) \cdot \mathrm{d}\, \operatorname{tr} L. \tag{6}$$

Explanation. The characteristic polynomial of the operator L $\chi(t) = \det(t \cdot \mathrm{Id} - L) = t^n - \sigma_1(x)t^{n-1} - \dots - \sigma_{n-1}(x)t - \sigma_n(x)$ is understood as a smooth function on M (i.e. w.r.t. local coordinates x) with t being a formal parameter. In particular, the differential of $\chi(t)$ is understood as $\mathrm{d}\,\chi(t) = 0 \cdot t^n - t^{n-1}\mathrm{d}\,\sigma_1(x) - \dots - t\mathrm{d}\,\sigma_{n-1}(x) - \mathrm{d}\,\sigma_n(x)$.

Proof.

We first notice that for any operator L (not necessarily Nijenhuis) and vector field η the following identity holds:

$$\mathcal{L}_\eta \det L = \operatorname{tr}\left(\widehat{L}\,\mathcal{L}_\eta L\right), \tag{7}$$

where $\widehat{L}$ denotes the adjugate matrix of L. Now suppose that L is a Nijenhuis operator and therefore $\mathcal{L}_{L\xi} L = L\mathcal{L}_\xi L$ for any vector field ξ. Replacing η with $L\xi$ in (7) and using the fact that $\widehat{L}\,L = \det L \cdot \mathrm{Id}$, we get:

$$\mathcal{L}_{L\xi} \det L = \operatorname{tr}\left(\widehat{L}\,\mathcal{L}_{L\xi} L\right) = \operatorname{tr}\left(\widehat{L}\,L\,\mathcal{L}_\xi L\right) = \det L \cdot \operatorname{tr} \mathcal{L}_\xi L = \det L \cdot \mathcal{L}_\xi \operatorname{tr} L,$$

as stated.
Finally, formula

$$L^*\left(\mathrm{d}\,\chi(t)\right) - t\cdot \mathrm{d}\,\chi(t) = \chi(t)\cdot \mathrm{d}\,\operatorname{tr} L$$

is obtained from

$$L^*\mathrm{d}\left(\det L\right) = \det L \cdot \mathrm{d}\,\operatorname{tr} L$$

by replacing L with the Nijenhuis operator $t\cdot \mathrm{Id} - L$ (here we think of t as a constant). □

Corollary

Let $\sigma_1, \dots, \sigma_n$ *be the coefficients of the characteristic polynomial*

$$\chi(t) = \det(t \cdot \mathrm{Id} - L) = t^n - \sigma_1\, t^{n-1} - \sigma_2\, t^{n-2} - \dots - \sigma_{n-1}\, t - \sigma_n$$

of a Nijenhuis operator L. *Then in any local coordinate system* $x_1, \dots, x_n$ *the following matrix relation hold:*

$$J(x)\, L(x) = S_\chi(x)\, J(x), \quad \textit{where } S_\chi(x) = \begin{pmatrix} \sigma_1(x) & 1 & & & \\ \sigma_2(x) & 0 & 1 & & \\ \vdots & \vdots & \ddots & \ddots & \\ \sigma_{n-1}(x) & 0 & \dots & 0 & 1 \\ \sigma_n(x) & 0 & \dots & 0 & 0 \end{pmatrix} \tag{8}$$

and $J(x)$ *is the Jacobi matrix of the collection of functions* $\sigma_1, \dots, \sigma_n$ *w.r.t. the variables* $x_1, \dots, x_n$.

Proof.

This matrix relation is equivalent to and can be easily obtained from (6), if we rewrite it as $L^*\big(\mathrm{d}\,\chi(t)\big) = -\chi(t) \cdot \mathrm{d}\,\sigma_1 + t \cdot \mathrm{d}\,\chi(t)$ and equate the terms of the same power in t. □

Almost obvious but important fact

Corollary

Assume that the coefficients $\sigma_1, \ldots, \sigma_n$ *of the characteristic polynomial of a Nijenhuis operator* L *are functionally independent at a point* $p \in M$ *(that is, their differentials* $\mathrm{d}\,\sigma_1(p), \ldots, \mathrm{d}\,\sigma_1(p)$ *are linearly independent). Then* L *can be uniquely reconstructed from* $\sigma_1, \ldots, \sigma_n$ *in a neighbourhood of* p*.*

More globally, if we have two Nijenhuis operators L_1 *and* L_2 *whose characteristic polynomials on* M *coincide and the coefficients of these polynomials are functionally independent almost everywhere on* M*, then* $L_1 = L_2$*.*

Proof.

Formula (8) allows us, in fact, to get an explicit expression for L in any local coordinate system $x_1, \ldots, x_n$ at those points where $\sigma_1, \ldots, \sigma_n$ are functionally independent. Namely, $L(x) = J^{-1}(x)\, S_\chi(x)\, J(x)$, which implies the statement. □

Eigenvalues of Nijenhuis operators

Proposition

Let L be a Nijenhuis operator and $\lambda(x)$ be a smooth eigenvalue of L. Then

$$(L - \lambda(x))^* \mathrm{d}\, \lambda(x) = 0. \tag{9}$$

Proof.

We will prove this relation at a point $p \in M$ such that: (i) $\lambda(x)$ has constant multiplicity k in $U(p)$, (ii) $\mathrm{d}\, \lambda(p) \neq 0$ and (iii) $\lambda(p) = 0$.
Because of (ii) we can choose local coordinates in in such a way that $\lambda(x) = x_n$. We need to show that $L^* \mathrm{d}\, x_n = 0$.
We know from (5) that $L^*(\mathrm{d}\ \det L) = \det L \cdot \mathrm{d}\ \mathrm{tr}\, L$.
Substituting $\det L = x_n^k f(x)$ and dividing by x_n^{k-1} we obtain:

$$L^* \left(k\, f(x) \mathrm{d}\, x_n + x_n \mathrm{d}\, f(x)\right) = x_n f(x) \cdot \mathrm{d}\ \mathrm{tr}\, R$$

Finally taking into account that $\lambda(p) = x_n(p) = 0$, we see that at this point $p \in M$:

$$L^* \left(k\, f(p) \mathrm{d}\, x_n\right) = 0$$

and, since $f(p) \neq 0$, the statement follows. □

Exercises

Exercise

Let $A(t)$ be a matrix whose entries are smooth functions of t. Prove that

$$\frac{\mathrm{d}}{\mathrm{d}\,t} \det A(t) = \operatorname{tr}\left(\widehat{A}(t)\frac{\mathrm{d}\,A(t)}{\mathrm{d}\,t}\right)$$

where $\widehat{A}$ denotes the adjugate matrix of A (transpose of the cofactor matrix). Recall that $\hat{A}\,A = A\,\hat{A} = \det A \cdot \mathrm{Id}$.

Exercise

Let $p_x(t) = t^n + a_1(x)t^{n-1} + \cdots + a_{n-1}(x)t + a_n(x)$ be a polynomial whose coefficients are smooth functions in $x \in \mathbb{R}^n$. Assume that at a point x_0, we have $p_{x_0}(t) = q_{x_0}(t)r_{x_0}(t)$, where q_{x_0} and r_{x_0} are polynomials with no common roots. Show that this factorisation can be uniquely extended to some neighbourhood $U(x_0)$, i.e., $p_x(t) = q_x(t)r_x(t)$ for any $x \in U(x_0)$ and the coefficients of $q_x(t)$ and $r_x(t)$ are smooth functions of x.

Exercise

Let L be a 2×2 Nijenhuis operator. Assume that $u = \operatorname{tr} L$, $v = \det L$ are independent functions. What is the matrix of L in the coordinate system u, v? Assume that the eigenvalues λ_1 and λ_2 of L are smooth independent functions. What is the matrix of L in the coordinate system λ_1, λ_2?