Outline of Lecture 3

- What is an analytic function $f(L)$ of an operator L?
- If L is Nijenhuis, then $f(L)$ is Nijenhuis
- Examples: projectors and complex structures
- Distributions and their integrals
- Frobenius theorem on integrability of distributions
- Splitting theorem

Analytic functions of matrices (operators)

We know from the previous lecture that any polynomial $p(L)$ of a Nijenhuis operator L is a Nijenhuis operator too. Our goal is to generalise this property to the case of real analytic functions $f(L)$.

What kind of function we are going to consider? Examples include polynomials, matrix exponent $\exp A = \mathrm{Id} + A + \frac{1}{2}A + \frac{1}{3!}A^3 + \dots$, $\sin A$, $\sinh A$ and many others.

General construction. Suppose a compact set $K \subseteq \mathbb{C}$, a function $f : K \to \mathbb{C}$ and an operator L (on M) satisfy the following assumptions:

(i) $\mathbb{C} \setminus K$ is connected (we do not require that K is connected);

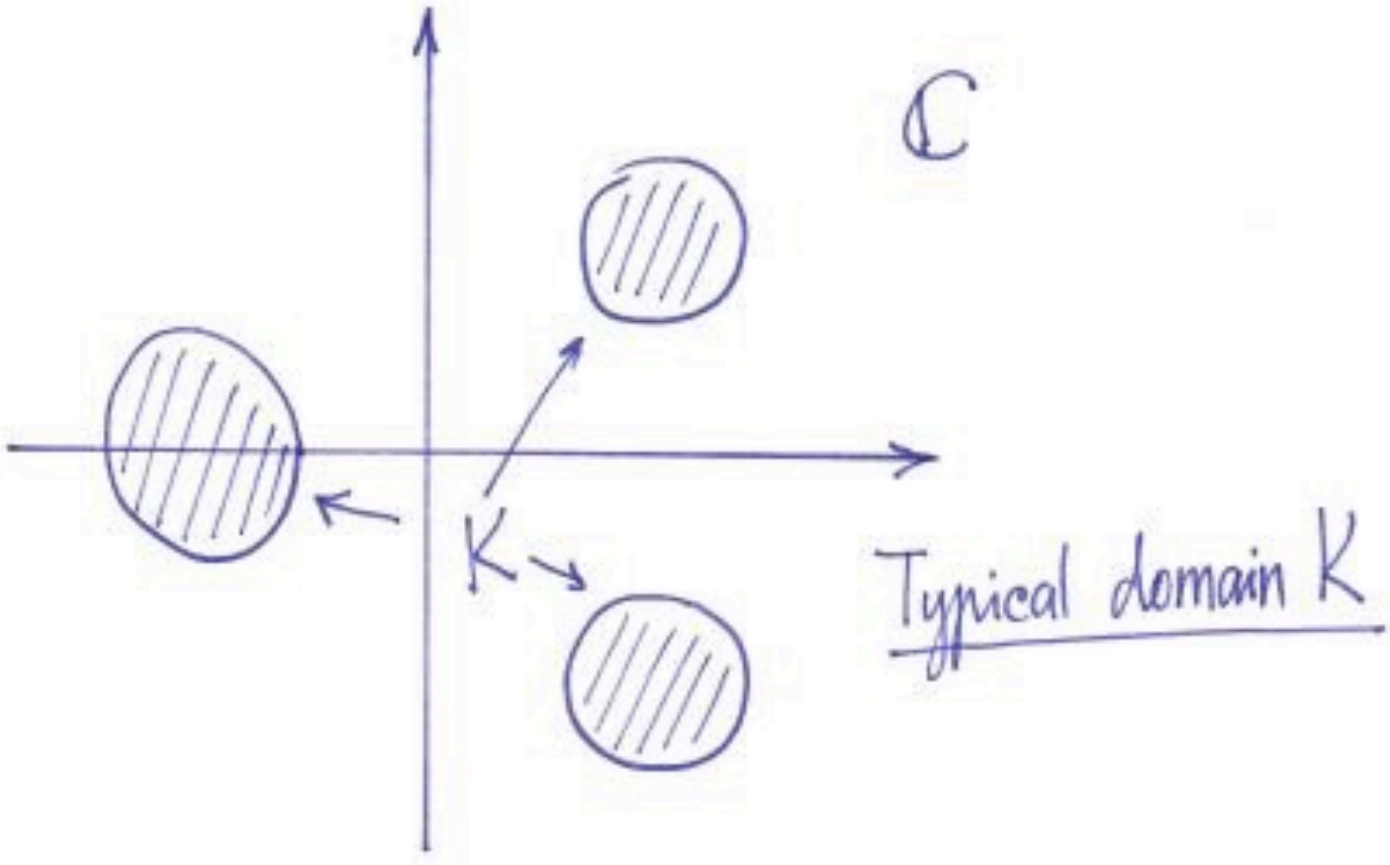
(ii) $f : K \to \mathbb{C}$ is a continuous function, and the restriction $f|_{\mathrm{int}(K)}$ is holomorphic;

(iii) K is symmetric with respect to the x-axes: for every $z \in K$ its conjugate $\bar{z}$ also belongs to K.

(iv) $f(z) = \bar{f}(\bar{z})$ for every $z \in K$;

(v) for every $x \in M$ we have $\mathrm{Spectrum}_x L \subset \mathrm{int}(K)$.

Figure



Analytic functions of matrices (operators)

Under the assumptions (i–v), one can naturally define an operator $f(L)$[1].

Mergelyan's theorem. The function f can be uniformly approximated by polynomials with real coefficients.

We define $f(L) = \lim_{k\to\infty} p_k(L)$.

One can show that the limit exists, is independent on the choice of the sequence p_k, smoothly depends on $x \in M^n$ (actually, the function $f(L)$ is analytic in the entries of L), and behaves as a $(1,1)$-tensor under coordinate transformations. Moreover, $\mathrm{Spectrum}_x f(L) = f(\mathrm{Spectrum}_x L)$.

Finally, since uniform convergence of holomorphic functions implies convergence of derivatives, we also have $\mathcal{L}_\xi\big(f(L)\big) = \lim_{k\to\infty} \mathcal{L}_\xi\big(p_k(L)\big)$ where $\mathcal{L}_\xi$ denotes the Lie derivative (in particular, a similar relation holds for partial derivatives).

Theorem

Let L be a Nienhuis operator. Then $f(L)$ is a Nijenhuis operator for any function $f : K \to \mathbb{C}$ satisfying assumptions (i)–(v).

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[1] N. J. Higham, *Functions of matrices. Theory and computation*, SIAM, Philadelphia, PA, 2008.

Example 1: Projectors

Example

Consider the characteristic polynomial $\chi(t)$ of L

$$\chi : M \times \mathbb{R} \to \mathbb{R}, \quad (x, t) \mapsto \chi(t) := \det\big(t \cdot \mathrm{Id} - L(x)\big).$$

We will say that a factorisation $\chi(t) = \chi_1(t) \cdot \chi_2(t)$ is *admissible*, if $\chi_1, \chi_2 : M \times \mathbb{R} \to \mathbb{R}$ are monic polynomials of degree ≥ 1 with smooth coefficients, which are coprime at every point $x \in M$ (i.e., have no common roots).

Consider the following two distributions $\mathcal{D}_1, \mathcal{D}_2$ on M:

$$\mathcal{D}_i = \mathrm{Ker}\, \chi_i(L) \ , \quad i = 1, 2, \tag{1}$$

where Ker denotes the kernel. It follows from Linear Algebra that

- $\mathcal{D}_1$ and $\mathcal{D}_2$ are L-invariant,
- $TM = \mathcal{D}_1 \oplus \mathcal{D}_2$.

This decomposition defines two natural projectors P_1 and P_2 onto the distributions $\mathcal{D}_1$ and $\mathcal{D}_2$ respectively. We now interpret them as analytic functions $P_1 = f_1(L)$ and $P_2 = f_2(L)$ satisfying assumptions (i)–(v).

Figure

$L : V \to V \qquad V = \bigoplus_{i=1}^{s} V_{\lambda_i} \qquad \lambda_i$ eigenvalue

$\{\lambda_1 \dots \lambda_p\} \qquad \{\lambda_{p+1}, \dots \lambda_s\}$ two groups of eigenvalues

V_{λ_i} generalised eigenspace

$$D_1 = \bigoplus_{i=1}^{p} V_{\lambda_i}, \quad D_2 = \bigoplus_{i=p+1}^{s} V_{\lambda_i}$$

$$V = D_1 \oplus D_2$$

Another interpretation of the subspaces D_1 and D_2

Projectors (continued)

Take a point $x \in M$ and consider the zeros $\lambda_1, \ldots, \lambda_r$ and $\lambda_{r+1}, \ldots, \lambda_n$ of the polynomials χ_1 and respectively χ_2 at this point. Choose $\varepsilon > 0$ and consider

$$K_1 := \bigcup_{i=1}^{r} B(\lambda_i, \varepsilon) \ , \quad K_2 := \bigcup_{i=r+1}^{n} B(\lambda_i, \varepsilon) \ , \quad K := K_1 \cup K_2,$$

where $B(z, \varepsilon) \subset \mathbb{C}$ denotes a closed ball of radius ε centered at $z \in \mathbb{C}$. Let ε be small enough so that $K_1 \cap K_2 = \emptyset$ and K satisfy assumptions (i) and (iii). In a small neighbourhood of x, condition (v) is obviously fulfilled.
Now take the (locally constant) function

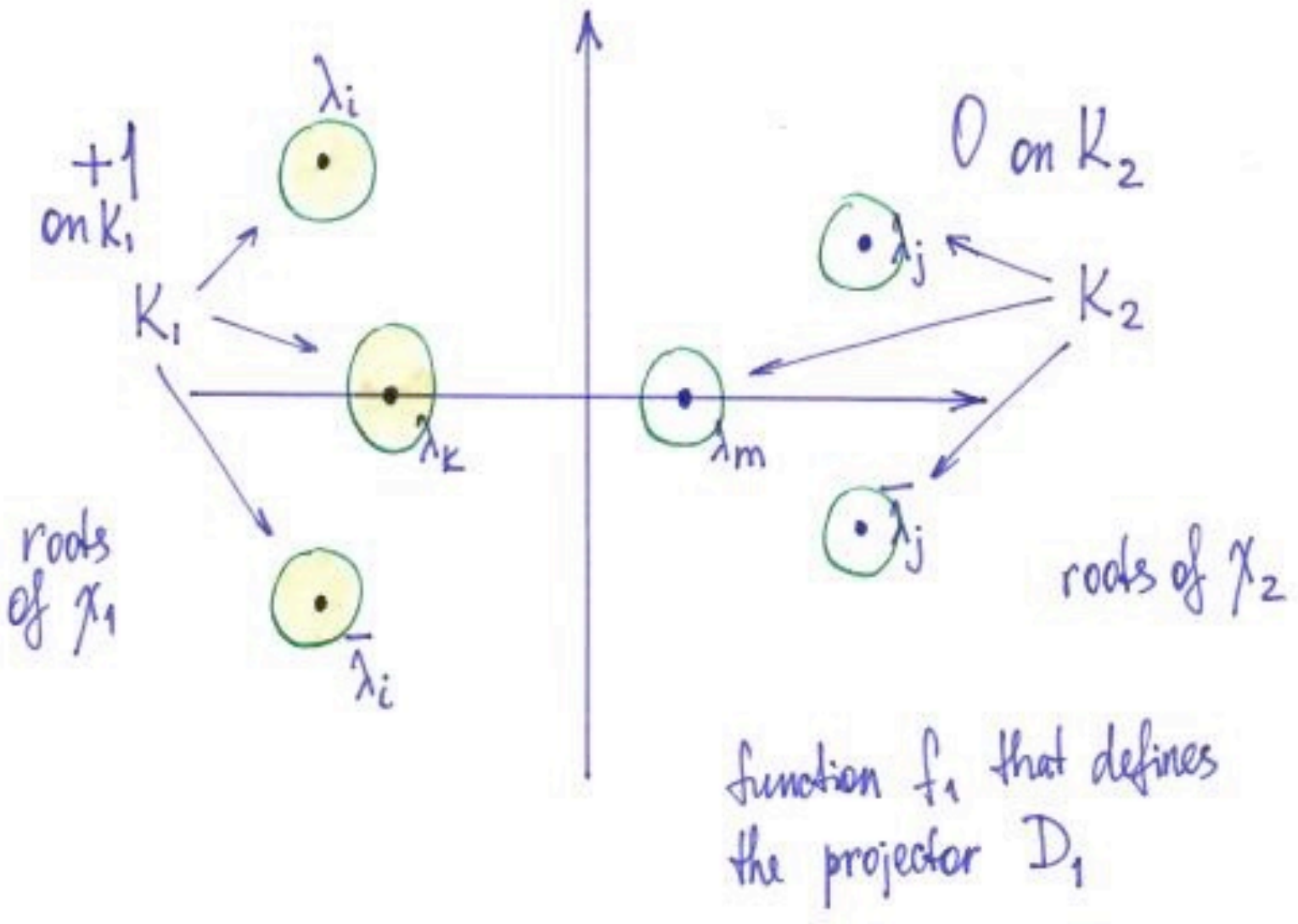
$$f_1 : K \to \mathbb{C}, \quad f_1(z) := \begin{cases} 1 & \text{for } z \in K_1, \\ 0 & \text{for } z \in K_2. \end{cases}$$

The function satisfies (ii, iv) and, therefore, $f_1(L)$ is well-defined.
It is easy to see that $f_1(L) = P_1$. Similarly, one construct a function f_2 such that $f_2(L) = P_2$.

Proposition

Let L be Nijenhuis. Then the projectors P_1 and P_2 are Nijenhuis also.

Figure


λ_i
+1 on K_1
0 on K_2
λ_j
K_1
K_2
λ_k
λ_m
λ_j
roots of χ_1
roots of χ_2
λ_i
function f_1 that defines the projector D_1

Example 2: Complex structure

Consider the following function on $\mathbb{C} \setminus \mathbb{R}$:

$$f(z) = \begin{cases} \mathrm{i}\,, & \text{if } \operatorname{Im} z > 0, \\ -\mathrm{i}\,, & \text{if } \operatorname{Im} z < 0. \end{cases}$$

This function is locally constant and holomorphic on $\mathbb{C} \setminus \mathbb{R}$. It is easy to see that all properties (i)–(iv) are fulfilled for any symmetric compact subset $K \subset \mathbb{C} \setminus \mathbb{R}$.

Proposition

Let L be an operator on a real manifold M with no real eigenvalues. Then the operator

$$J = f(L) \tag{2}$$

is well defined on M and satisfies the following properties:

1. J is smooth;
2. $J^2 = -\mathrm{Id}$, i.e. J is an almost complex structure on M;
3. $JL = LJ$, i.e. L is a complex linear operator w.r.t. J;
4. if L is Nijenhuis, then J is Nijenhuis too and hence is a complex structure on M.

Distributions and their integrals

Let $\mathcal{D}_x \subset T_x M^n$ be a subspace of dimension k (for each point $x \in M^n$). If D_x smoothly depends on x, we say that $\mathcal{D} = \{\mathcal{D}_x,\ x \in M^n\}$ is a k-dimensional distribution on M^n.

(A) Let $\xi_1, \dots, \xi_k$ be vector fields that are linearly independent at each point $x \in M^n$. Then they generate the natural distribution

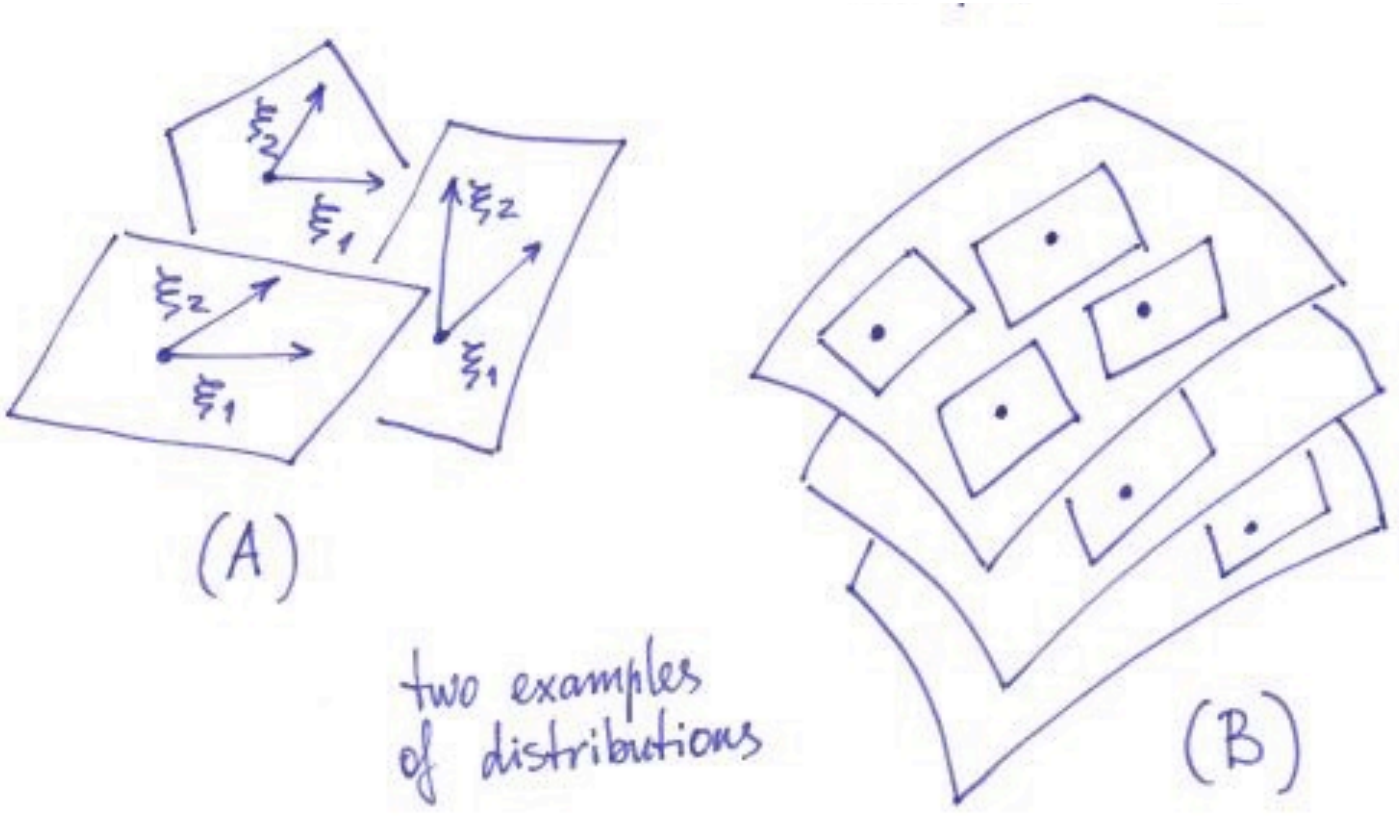
$$\mathcal{D} = \operatorname{Span}(\xi_1, \dots, \xi_k).$$

Locally, every distribution can be presented in this way.

(B) Let $f_1, \dots, f_{n-k} : M^n \to \mathbb{R}$ be independent functions such that $\mathrm{d}\, f_1(x), \dots, \mathrm{d}\, f_{n-k}(x)$ are linearly independent at each point. Equivalently, these functions define submersion $F : M^n \to \mathbb{R}^{n-k}$. As a distribution, we can take $\mathcal{D}_x = \operatorname{Ker} \mathrm{d}\, F(x) \subset T_x M$. Notice that $\mathcal{D}_x$ are tangent spaces $T_x N$ to the fibers, i.e., common level sets $N = \{f_1(x) = a_1, \dots, f_{n-k}(x) = a_{n-k}\} = F^{-1}(a)$ and $\xi(f_i) = \mathcal{L}_\xi(f_i) = 0$ for any vector field $\xi \in \mathcal{D}$.

Distribution $\mathcal{D}$ is called integrable if locally it can be presented as in (B). In other words, we can find $n - k$ independent functions $f_1, \dots, f_{n-k}$ (integrals of $\mathcal{D}$) such that $\xi(f_i) = 0$ for all $\xi \in \mathcal{D}$ and $i = 1, \dots, n - k$.

Figure



Two Frobenius integrability theorem

Theorem (Frobenuis 1)

Let $\mathcal{D} = \mathrm{Span}(\xi_1, \dots, \xi_k)$ be a distribution generated by k linearly independent vector fields $\xi_1, \dots, \xi_k$. This distribution is integrable if and only if

$$[\xi_i, \xi_j] \in \mathcal{D} \qquad \text{for all } i, j = 1, \dots, k,$$

in other words, $[\xi_i, \xi_j] = \sum_k f_{ij}^k(x)\xi_k$ for some functions $f_{ij}^k(x)$.

Theorem (Frobenius 2)

Let $\xi_1, \dots, \xi_k$ be linearly independent vector fields, $k \leq n = \dim M$. Then the following conditions are equivalent:

(i) *there exists a local coordinate system $x^1, \dots, x^n$ such that $\xi_i = \partial_{x^i}$, i.e., $\xi_i = (0, \dots, 0, \underset{i}{1}, 0, \dots, 0)$, $i = 1, \dots, k$;*

(ii) *these vector fields pairwise commute, i.e., $[\xi_i, \xi_j] = 0$ for all $i, j = 1, \dots, k$.*

Frobenius 2 (commuting vector fields)

Consider the flow ϕ_ξ^t generated by a vector filed ξ (see Figure).
Properties:

- $\frac{d}{dt}\big|_{t=0} \phi_\xi^t(p) = \xi(p)$;
- $\phi_\xi^{t+s}(p) = \phi_\xi^t \circ \phi_\xi^s(p)$;
- Locally: $[\xi, \eta] = 0$ if and only if the corresponding flows commute, i.e. $\phi_\xi^t \circ \phi_\eta^s(p) = \phi_\eta^s \circ \phi_\xi^t(p)$.

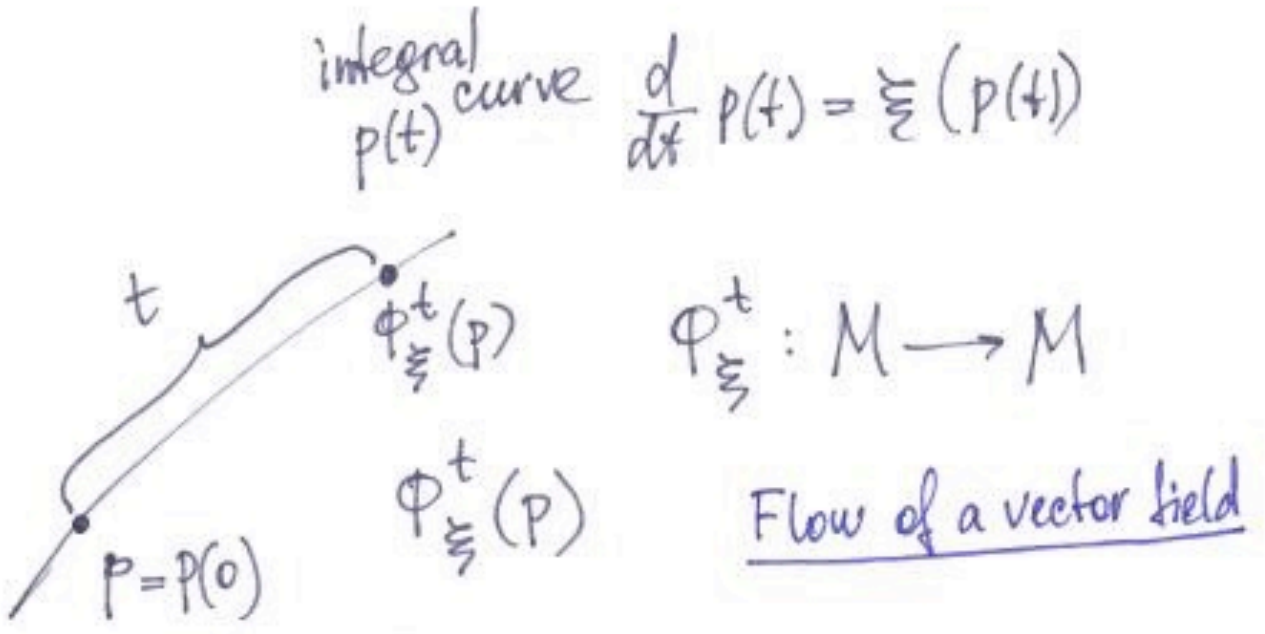
Now take a point $p \in M^n$ and choose an arbitrary submanifold $N^{n-k} \subset M^n$ through p which is transversal to $\mathrm{Span}(\xi_1, \dots, \xi_k)$.
Let $y^1, \dots, y^{n-k}$ be local coordinates on N, and $p = (0, \dots, 0)$.
Consider the following map

$$\Phi : U \to M^n, \quad U = U(0) \subset \mathbb{R}^n = \mathbb{R}^k_{(x^1,\dots,x^k)} \times \mathbb{R}^{n-k}_{(y^1,\dots,y^{n-k})},$$

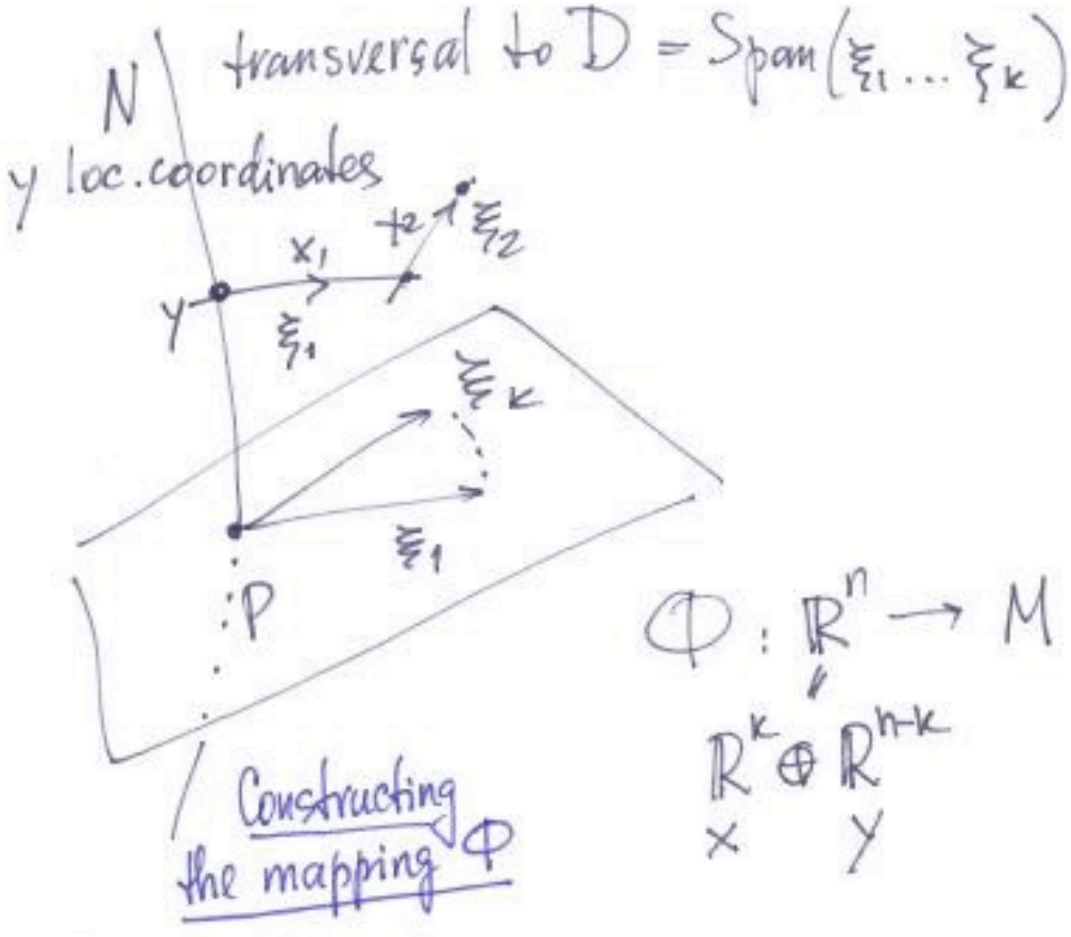
$$\Phi(x^1, \dots, x^k, y^1, \dots, y^{n-k}) = \phi_{\xi_1}^{x^1} \circ \phi_{\xi_2}^{x^2} \circ \cdots \circ \phi_{\xi_k}^{x^k}(y), \quad y = (y^1, \dots, y^{n-k}) \in N.$$

Notice that the differential $d\Phi : T_0\mathbb{R}^n \to T_pM^n$ is an isomorphism.
Therefore, Φ is a local diffeomorphism in some neighborhood of $0 \in \mathbb{R}^n$.
This allows us to consider $x^1, \dots, x^k, y^1, \dots y^{n-k}$ as local coordinates near $p \in M^n$.

Figure


integral curve
$p(t)$
$\frac{d}{dt} p(t) = \xi(p(t))$
t
$\varphi_\xi^t(p)$
$\varphi_\xi^t : M \longrightarrow M$
$\varphi_\xi^t(p)$
$p = p(0)$
Flow of a vector field

Figure


N
transversal to D = Span($\xi_1 \dots \xi_k$)
y loc. coordinates
x_1
x_2
ξ_2
y
ξ_1
ξ_k
ξ_1
P
$\Phi : \mathbb{R}^n \to M$
$\mathbb{R}^k \oplus \mathbb{R}^{n-k}$
x y
Constructing the mapping Φ

Frobenius 2 (continued)

Let us show that this coordinate systems satisfies the required property $\xi_i = \partial_{x^i}$ (we will need to re-denote $y^i = x^{k+i}$).

Indeed, by definition, the basis vector field ∂_{x^i} (at the point $\Phi(x, y)$ with coordinates $(x^1, \dots, x^k, y^1, \dots y^{n-k})$) is defined by

$$\partial_{x^i} = \left.\frac{d}{dt}\right|_{t=0} \Phi(x^1, \dots, x^i + t, \dots, x^k, y^1, \dots, y^{n-k})$$

Let us compute it:

$$= \left.\frac{d}{dt}\right|_{t=0} \phi^{x^1}_{\xi_1} \circ \cdots \circ \phi^{x^i+t}_{\xi_i} \circ \cdots \circ \phi^{x^k}_{\xi_k}(y) = \left.\frac{d}{dt}\right|_{t=0} \phi^{x^1}_{\xi_1} \circ \cdots \circ \phi^{t}_{\xi_i} \circ \phi^{x^i}_{\xi_i} \circ \cdots \circ \phi^{x^k}_{\xi_k}(y) =$$

$$\left.\frac{d}{dt}\right|_{t=0} \phi^{t}_{\xi_i} \circ \phi^{x^1}_{\xi_1} \circ \cdots \circ \phi^{x^i}_{\xi_i} \circ \cdots \circ \phi^{x^k}_{\xi_k}(y) = \left.\frac{d}{dt}\right|_{t=0} \phi^{t}_{\xi_i}\big(\Phi(x, y)\big) = \xi_i\big(\Phi(x, y)\big)$$

as was to be proved.

Proof of Frobenius 1

$\Leftarrow$ Easy.

$\Rightarrow$ We assume that $\xi_1, \ldots, \xi_k$ satisfy $[\xi_i, \xi_j] = \sum f_{ij}^k(x)\xi_k$.
Consider the matrix composed of the components of ξ_i

$$B = \begin{pmatrix} \xi_1^1 & \cdots & \xi_1^k & \xi_1^{k+1} & \cdots & \xi_1^n \\ \xi_2^1 & \cdots & \xi_2^k & \xi_2^{k+1} & \cdots & \xi_2^n \\ & & & & & \\ \xi_k^1 & \cdots & \xi_k^k & \xi_k^{k+1} & \cdots & \xi_k^n \end{pmatrix} \text{ and assume w.l.o.g. } \det \begin{pmatrix} \xi_1^1 & \cdots & \xi_1^k \\ \xi_2^1 & \cdots & \xi_2^k \\ & & \\ \xi_k^1 & \cdots & \xi_k^k \end{pmatrix} \neq 0.$$

Then there exists a matrix A such that

$$AB = \begin{pmatrix} 1 & 0 & \ldots & 0 & \eta_1^{k+1} & \ldots & \eta_1^n \\ 0 & 1 & \ldots & 0 & \eta_2^{k+1} & \ldots & \eta_2^n \\ & & \ddots & & & & \\ 0 & 0 & \ldots & 1 & \eta_k^{k+1} & \ldots & \eta_k^n \end{pmatrix} \begin{matrix} \leftarrow \eta_1 \\ \leftarrow \eta_2 \\ \vdots \\ \leftarrow \eta_n \end{matrix}$$

These 'new' vector fields $\eta_1, \ldots, \eta_k$ generate the same distribution and satisfy $[\eta_i, \eta_j] = \sum_k \tilde{f}_{ij}^k(x)\eta_k$. However, one can check that this relation, for the vector fields of this kind, in fact implies $[\eta_i, \eta_j] = 0$. Hence, the statement follows from 'Frobenius 2'.

Figure

$$\eta_1 = (1\ 0 \dots 0\ \eta_1^{k+1} \dots \eta_1^n)$$
$$\eta_2 = (0\ 1 \dots 0\ \eta_2^{k+1} \dots \eta_2^n)$$

if $[\eta_1, \eta_2]$ = lin. comb. of $\eta_1 \dots \eta_k$, then

$$[\eta_1, \eta_2] = 0$$

Example

$$\eta_1 = (1, 0, f)$$
$$\eta_2 = (0, 1, g)$$

$$[\eta_1, \eta_2] = [\partial_{x_1} + f\partial_{x_3}, \partial_{x_2} + g\partial_{x_3}] =$$
$$= \frac{\partial g}{\partial x_1}\partial_{x_3} + f\frac{\partial g}{\partial x_3}\partial_{x_3} - \frac{\partial f}{\partial x_2}\partial_{x_3} - g\frac{\partial f}{\partial x_3}\partial_{x_3} =$$
$$= (\dots)\partial_{x_3} = \text{lin. comb. of } \eta_1 \text{ and } \eta_2 =$$
$$= c_1(\partial_{x_1} + f\partial_{x_3}) + c_2(\partial_{x_2} + g\partial_{x_3})$$
$$\Rightarrow c_1 = c_2 = 0$$

Property of the image of a Nijenhuis operator

Corollary

Let L be a Nijenhuis operator of rank $k < n$. Consider the k-dimensional distribution $\mathcal{D} = \mathrm{Image}(L)$. This distribution is integrable.

Proof.

The image of L is generated by the vector fields of the form $L\xi$, where ξ runs over all vector fields. According to Frobenuis Theorem, we need to show that for any two vector fields $L\xi$ and $L\eta$, their commutator $[L\xi, L\eta]$ belongs to $\mathrm{Image}(L)$. But this follows immediately from the definition:

$$L^2[\xi,\eta] - L[L\xi,\eta] - L[\xi,L\eta] + [L\xi,L\eta] = 0, \quad \text{or, equivalently:}$$

$$\begin{aligned}[L\xi,L\eta] &= -L^2[\xi,\eta] + L[L\xi,\eta] + L[\xi,L\eta] \\ &= L\Big(-L[\xi,\eta] + [L\xi,\eta] + [\xi,L\eta]\Big) \in \mathrm{Image}(L).\end{aligned}$$

□

Remark. Along with the image of L, we may consider the kernel $\mathrm{Ker}(L)$ as another important distribution. Is it always integrable? The answer is negative: there exist Nijenhuis operators with non-integrable kernels.

Admissible factorisations

Consider a point $p \in M$ and let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues (possibly complex) of $L(p)$ counted with multiplicities. We divided them into two groups $\{\lambda_1, \ldots, \lambda_r\}$ and $\{\lambda_{r+1}, \ldots, \lambda_n\}$ in such a way that equal eigenvalues as well as complex conjugate pairs belong to the same group.

Then, at the point p we have an admissible factorisation of the characteristic polynomial $\chi_{L(p)}(t) = \chi_1(t)\,\chi_2(t)$ (see Example 1), where

$$\chi_1 = (t - \lambda_1)\ldots(t - \lambda_r), \quad \chi_2 = (t - \lambda_{r+1})\ldots(t - \lambda_n).$$

By continuity, this partition of eigenvalues can be extended to a certain neighbourhood $U(p)$, so that locally we have an admissible factorisation of the characteristic polynomial of L:

$$\chi_{L(p)}(t) = \chi_1(t)\,\chi_2(t).$$

The fact that the coefficients of $\chi_1(t)$ and $\chi_2(t)$ are smooth on $U(p)$ easily follows from the Implicit Function Theorem (Exercise in Lecture 2). Notice that we do not assume that $p \in M$ is generic, i.e., some eigenvalues may collide at p.

Splitting Theorem

Following Example 1 (Projectors), we consider the distributions $\mathcal{D}_i = \operatorname{Ker} \chi_i(L)$ $(i = 1, 2)$ that provide a natural decomposition of the tangent bundle $TM = \mathcal{D}_1 \oplus \mathcal{D}_2$.

Theorem

Let $\chi_L(t) = \chi_1(t)\,\chi_2(t)$ be an admissible factorisation of the characteristic polynomial of a Nijenhuis operator L in a neighbourhood of a point $p \in M$. Then the distributions $\mathcal{D}_i = \operatorname{Ker} \chi_i(L)$ are both integrable.

Moreover, in any adapted coordinate system $(x_1, \dots, x_r, y_1, \dots, y_{n-r})$ (i.e., such that $\mathcal{D}_1$ is generated by the vectors ∂_{x_i}, and $\mathcal{D}_2$ is generated by the vectors ∂_{y_j}) the operator L takes the block-diagonal form

$$L(x, y) = \begin{pmatrix} L_1(x) & 0 \\ 0 & L_2(y) \end{pmatrix}, \tag{3}$$

where the entries of the $r \times r$ matrix L_1 depend on the x-variables only, and the entries of the $(n - r) \times (n - r)$ matrix L_2 depend on the y-variables only. In other words, L splits into a direct sum of two Nijenhuis operators: $L(x, y) = L_1(x) \oplus L_2(y)$.

Proof.

We have $TM = \mathcal{D}_1 \oplus \mathcal{D}_2$. Consider the projectors P_1 and P_2 onto the distributions $\mathcal{D}_1$ and $\mathcal{D}_2$ so that $\mathcal{D}_i = \mathrm{Image}(P_i)$. Since P_i is a Nijenhuis operator (Example 1), then $\mathcal{D}_i = \mathrm{Image}(P_i)$ is integrable.

The integrability of $\mathcal{D}_1$ and $\mathcal{D}_2$ is equivalent to the existence of an adapted local coordinate system (x, y) such that $\partial_{x_1}, \ldots, \partial_{x_r}$ generate $\mathcal{D}_1$ and $\partial_{y_1}, \ldots, \partial_{y_{n-r}}$ generates $\mathcal{D}_2$. Since these distributions are both L-invariant, the operator L in this coordinate system has a block diagonal form:

$$L(x, y) = \begin{pmatrix} L_1(x, y) & 0 \\ 0 & L_2(x, y) \end{pmatrix}.$$

Without loss of generality we may assume that $\det L \neq 0$, otherwise we can (locally) replace L by $L + c \cdot \mathrm{Id}$ where c is an appropriate constant. Notice that the operator

$$LP_1 = \begin{pmatrix} L_1(x, y) & 0 \\ 0 & 0 \end{pmatrix},$$

is an analytic function of L as a product of two analytic functions, namely L itself and $P_1 = f_1(L)$ introduced in Example 1. Hence LP_1 has zero Nijenhuis torsion.

Thus, for $u = \partial_{y_\alpha} \in D_2 = \operatorname{Ker} P_1$ we have

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \mathcal{L}_{LP_1 u}(LP_1) - LP_1 \mathcal{L}_u(LP_1) = \text{(use that } LP_1 u = 0\text{)}$$

$$LP_1 \mathcal{L}_u(LP_1) =$$

$$\begin{pmatrix} L_1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \partial_{y_\alpha} L_1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} L_1 \partial_{y_\alpha} L_1 & 0 \\ 0 & 0 \end{pmatrix}$$

Since L_1 is non-degenerate we conclude that $\partial_{y_\alpha} L_1 = 0$, i.e., $L_1 = L_1(x)$. Similarly, $L_2 = L_2(y)$, as needed.

To compare with Poisson Geometry: Weinstein Splitting Theorem

One the fundamental results in Poisson Geometry is

Weinstein Splitting Theorem.
Every Poisson manifold splits locally (i.e., in a neighbourhood of a point $q \in M$) into a product of a symplectic manifold and a Poisson manifold that has rank zero at the given point.

In terms of matrices:

Let $P = \left(p^{ij}\right)$ be a Poisson tensor in a neighbourhood of a point q.
Then we can always find a local coordinate system (x, y) in which P takes the block-diagonal form

$$P(x, y) = \begin{pmatrix} P_1(x) & 0 \\ 0 & P_2(y) \end{pmatrix}.$$

such that $\det P_1 \neq 0$ and $P_2|_q = 0$.

Theorem

Assume that the spectrum of a Nijenhuis operator L at a point $p \in M$ consists of k real (distinct) eigenvalues $\lambda_1, \dots, \lambda_k$ with multiplicities $\mathsf{m}_1, \dots, \mathsf{m}_k$ respectively and s pairs of complex (non-real) conjugate eigenvalues $\mu_1, \bar{\mu}_1, \dots, \mu_s, \bar{\mu}_s$ of multiplicities $\mathsf{l}_1, \dots, \mathsf{l}_s$. Then in a neighbourhood of $p \in M$ there exists a local coordinate system

$$x_1 = (x_1^1 \dots x_1^{\mathsf{m}_1}), \dots, x_k = (x_k^1 \dots x_k^{\mathsf{m}_k}), \quad u_1 = (u_1^1 \dots u_1^{2\mathsf{l}_1}), \dots, u_s = (u_s^1 \dots u_s^{2\mathsf{l}_s}),$$

in which L takes the following block-diagonal form

$$L = \begin{pmatrix} L_1(x_1) & & & & & \\ & \ddots & & & & \\ & & L_k(x_k) & & & \\ & & & L_1^{\mathbb{C}}(u_1) & & \\ & & & & \ddots & \\ & & & & & L_s^{\mathbb{C}}(u_s) \end{pmatrix}$$

where each block depends on its own group of variables and is a Nijenhuis operator w.r.t. these variables. In other words, every Nijenhuis operator L locally splits into a direct sum of Nijenhuis operators $L = L_1 \oplus L_2 \oplus \dots$ each of which at the point $p \in M$ has either a single real eigenvalue or a single pair of complex eigenvalues.

Exercises

Exercise

Is $f(L) = L^{1/3}$ a real analytic function on the set of real matrices L with $\det L \neq 0$?

Exercise

It is well-known that every linear operator L can be presented as $L = S + N$, where S is semisimple, N is nilpotent and $NS = SN$. This decomposition is unique and known as Jordan-Chevalley decomposition (https://en.wikipedia.org/wiki/Jordan-Chevalley_decomposition). Both S and N can be treated as functions of L. Are these functions real analytic?

Exercise

Let $L(x)$ be a Nojenhuis operator with a single eigenvalue $\lambda(x)$. Prove that $\lambda(x)$ is a smooth function and consider the decomposition $L(x) = \lambda(x)\,\mathrm{Id} + N(x)$. Is $N(x)$ a Nijenhuis operator?

Exercise

Give an example of a Nijenhuis operator with a non-integrable kernel. Prove that if $u, v \in \operatorname{Ker} L$, then $[u, v] \in \operatorname{Ker} L^2$. Let $\lambda(x)$ be a smooth eigenvalue of L. Prove that the distribution $\operatorname{Image}(L - \lambda(x)\mathrm{Id})$ is integrable.