Nijenhuis Geometry

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Lecture 4:
Nijenhuis operators in dimension 2 and Examples

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Spring 2021

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Outline of Lecture 4

- Simplifying Nijenhuis condition in dimension 2.
- Various examples
- Reconstructing L from $\operatorname{tr} L$ and $\det L$
- Characteristic map and admissible singularities
- Operators with complex eigenvalues, canonical complex structure
- "Portraits" of Nijenhuis operators
- Jordan block in dimension 2

Nijenhuis condition in dimension 2

Let L be a operator on M^2 . Fix local coordinates x, y. In these coordinates L can be written as a matrix

$$L = \begin{pmatrix} a(x,y) & b(x,y) \\ c(x,y) & d(x,y) \end{pmatrix}.$$

Theorem

L is a Nijenhuis operator if and only if the following condition holds:

$$\mathrm{d}\ \det L = \mathrm{d}\ \mathrm{tr}\, L \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \tag{1}$$

where the differential $\mathrm{d}\, f$ of a function f is treated as row-vector (f_x, f_y).

Remark. Notice that $\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ is the adjugate matrix of L. Thus we can rewrite the above relation in the form

$$\mathrm{d}\ \det L = (\mathrm{adj}\, L)^* \mathrm{d}\ \mathrm{tr}\, L. \tag{2}$$

Proof. Recall the general property of Nijenhuis operators:
$L^* \mathrm{d}\ \det L = \det L \cdot \mathrm{d}\ \mathrm{tr}\, L$, implying $\mathrm{d}\ \det L = \det L \cdot (L^{-1})^* \mathrm{d}\ \mathrm{tr}\, L$ which is equivalent to (2) as $\mathrm{adj}\, L = \det L \cdot L^{-1}$.

We use "tensor style" computations below, so we denote the components of L by L^i_j and set $x = x^1, y = x^2$. Vanishing of the Nijenhuis torsion in dimension 2 gives two PDEs $(\mathcal{N}_L)^1_{12} = 0, (\mathcal{N}_L)^2_{12} = 0$.
The first one in local coordinates has form

$$0 = (\mathcal{N}_L)^1_{12} = \frac{\partial L^1_2}{\partial x^1} L^1_1 + \frac{\partial L^1_2}{\partial x^2} L^2_1 - \frac{\partial L^1_1}{\partial x^1} L^1_2 - \frac{\partial L^1_1}{\partial x^2} L^2_2 - \frac{\partial L^1_2}{\partial x^1} L^1_1 - \frac{\partial L^2_2}{\partial x^1} L^1_2 + \frac{\partial L^1_1}{\partial x^2} L^1_1 + \frac{\partial L^2_1}{\partial x^2} L^1_2$$

The first and the fifth terms in this equation cancel out. We add $\frac{\partial L^2_2}{\partial x^2} L^1_1$ to the right side and subtract it. Regrouping the terms we get

$$\begin{aligned} 0 = {} & \frac{\partial}{\partial x^2}\Big(- L^1_1 L^2_2 + L^2_1 L^1_2\Big) \\ & - L^1_2\Big(\frac{\partial L^1_1}{\partial x^1} + \frac{\partial L^2_2}{\partial x^1}\Big) + L^1_1\Big(\frac{\partial L^1_1}{\partial x^2} + \frac{\partial L^2_2}{\partial x^2}\Big). \end{aligned} \tag{3}$$

In a similar way the equation $(\mathcal{N}_L)^2_{12} = 0$ can be rewritten in the form

$$\begin{aligned} 0 = {} & \frac{\partial}{\partial x^1}\Big(- L^1_1 L^2_2 + L^2_1 L^1_2\Big) \\ & L^2_2\Big(\frac{\partial L^1_1}{\partial x^1} + \frac{\partial L^2_2}{\partial x^1}\Big) - L^2_1\Big(\frac{\partial L^1_1}{\partial x^2} + \frac{\partial L^2_2}{\partial x^2}\Big). \end{aligned} \tag{4}$$

It remains to notice that $L^1_1 L^2_2 - L^2_1 L^1_2 = \det L$ and $L^1_1 + L^2_2 = \operatorname{tr} L$.
Equations (3) and (4) are equivalent to (1). $\square$

$$L = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \operatorname{adj} L = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \quad \operatorname{tr} L = a+d, \quad \det L = ad-bc$$

$$L^2[\xi,\eta] - L[L\xi,\eta] - L[\xi,L\eta] + [L\xi,L\eta] = 0$$

$$\xi = \partial_x, \quad \eta = \partial_y \qquad [\partial_x, \partial_y] = 0$$

$$-L[a\partial_x + c\partial_y, \partial_y] - L[\partial_x, b\partial_x + d\partial_y] + [a\partial_x + c\partial_y, b\partial_x + d\partial_y] = 0$$

$$\underbrace{-L(-a_y\partial_x - c_y\partial_y) - L(b_x\partial_x + d_x\partial_y)}_{\downarrow} + a(b_x\partial_x + d_x\partial_y) + c(b_y\partial_x + d_y\partial_y) - b(a_x\partial_x + c_x\partial_y) - d(a_y\partial_x + c_y\partial_y)$$

$$(a_y - b_x)(a\partial_x + c\partial_y) + (c_y - d_x)(b\partial_x + d\partial_y)$$

= collecting terms with ∂_x and ∂_y

$$= \underbrace{(a_y a - b_x a + c_y b - d_x b + a b_x + c b_y - b a_x - d a_y)}_{=0} \cdot \partial_x +$$

$$\underbrace{(a_y c - b_x c + c_y d - d_x d + a d_x + c d_y - b c_x - d c_y)}_{=0} \cdot \partial_y$$

We compare these relations with $d(\det L) = d(\operatorname{tr} L) \cdot \operatorname{adj} L$

$$((ad-bc)_x, (ad-bc)_y) = (a_x + d_x, a_y + d_y)\begin{pmatrix} d & b \\ -c & a \end{pmatrix}$$

$$(a_x d + d_x a - b_x c - c_x b, \underbrace{a_y d + d_y a - b_y c - c_y b}) = (a_x d + d_x d - a_y c - d_y c, \underbrace{-b a_x - b d_x + a a_y + a d_y})$$

equal

same condition

same condition

$$a_y d + d_y a - b_y c - c_y b + b a_x + b d_x - a a_y - a d_y = 0$$

$$a_x d + d_x a - b_x c - c_x b - a_x d - d_x d + a_y c + d_y c = 0$$

Corollaries/Examples

Proposition

$$L = \begin{pmatrix} g(y) & f(x,y) \\ 0 & g(y) \end{pmatrix},$$

where $f(x,y)$ and $g(y)$ are arbitrary smooth functions of two and one variables respectively, is a Nijenhuis operator.

Proposition

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Corollaries/Examples

Proposition

Let L be an operator (in dimension 2) with constant eigenvalues (equivalently, with constant trace and determinant). Then L is a Nijenhuis operator.

Proposition

Assume that $\operatorname{tr} L$ is constant (in dimension 2), then $\det L$ is constant too and L is a Nijenhuis operator.

Proposition

Every nilpotent operator in dimension 2 is Nijenhuis.

Example. In dimension > 2, these statements are false.

$$L = \begin{pmatrix} 0 & 1 & 0 \\ -y & 0 & 1 \\ 0 & y & 0 \end{pmatrix} \quad \text{is nilpotent, but } \mathcal{N}_L \neq 0.$$

Indeed, the image of L is generated by two vector fields ∂_y and $\partial_x + y\partial_z$, but the distribution $\operatorname{Span}(\partial_y, \partial_x + y\partial_z)$ is not integrable.

Reconstructing L from $\det L$ and $\operatorname{tr} L$

Let $\operatorname{tr} L$ and $\det L$ be independent almost everywhere (on a dense open subset). Then L can be reconstructed from the identity (See Lecture 2)

$$\begin{pmatrix} v_x & v_y \\ u_x & u_y \end{pmatrix} L = \begin{pmatrix} v & 1 \\ u & 0 \end{pmatrix} \begin{pmatrix} v_x & v_y \\ u_x & u_y \end{pmatrix},$$

where u and v are the coefficients of the characteristic polynomial

$$\chi_L(t) = \det(t \cdot \mathrm{Id} - L) = t^2 - vt - u, \qquad v = \operatorname{tr} L, \quad -u = \det L.$$

We can also rewrite the above relation in the form

$$L = \frac{1}{v_x u_y - u_x v_y} \begin{pmatrix} u_x & -v_y \\ -u_x & v_y \end{pmatrix} \begin{pmatrix} v & 1 \\ u & 0 \end{pmatrix} \begin{pmatrix} v_x & v_y \\ u_x & u_y \end{pmatrix} \tag{5}$$

If the differentials $\mathrm{d} \operatorname{tr} L$ and $\mathrm{d} \det L$ are linearly independent, i.e., $v_x u_y - u_x v_y \neq 0$, then this formula defines a good Nijenhuis operator. Indeed, at such a point we can take consider $v = \operatorname{tr} L$ and $u = -\det L$ as local coordinates, i.e., we simply set $v = x$ and $u = y$, then the above formula immediately gives

$$L = \begin{pmatrix} x & 1 \\ y & 0 \end{pmatrix}.$$

Problem of "admissible singularities"

However, at those points where $v_x u_y - u_x v_y = 0$, we have to "divide by zero" which leads us to a non-trivial problem of "admissible singularities". Consider the characteristic mapping

$$\Phi : M^2 \to \mathbb{R}^2, \qquad p \overset{\Phi}{\mapsto} \big(\operatorname{tr} L(p), \det L(p)\big).$$

Recall that a point p is said to be singular for Φ if $\operatorname{rank} \mathrm{d}\,\Phi(x) < 2$ or, equivalently $v_x u_y - u_x v_y = 0$. Consider an abstract characteristic map Φ and assume that its non-singular points form an open dense subset.

Definition

We will say that a singular point $p \in M^2$ is admissible if formula (5) defines a smooth operator L (i.e. L can be extended, by continuity, to singular points and the extended operator is smooth).

Problem. Describe all admissible singularities.

Even series of examples of such singularities would be very interesting. Let us discuss some of them.

Remark. Notice that we want to describe these singularities up to coordinate transformations.

Important case when $\mathrm{d}\ \mathrm{tr}\, L \neq 0$

Let us assume that $\operatorname{rank} \mathrm{d}\,\Phi(p) = 1 \neq 0$, for instance, $\mathrm{d}\ \mathrm{tr}\, L(p) \neq 0$.
Then we can choose local coordinates (x, y) in such a way that $y = \mathrm{tr}\, L$.

Denote $D = \det L$. To describe $L(x, y) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ we use formula (1):

$$\mathrm{d}\ \det L = \mathrm{d}\ \mathrm{tr}\, L \cdot \operatorname{adj} L, \quad \text{i.e.} \quad (D_x, D_y) = (0, 1) \begin{pmatrix} d & -b \\ -c & a \end{pmatrix},$$

This gives $D_x = -c$, $D_y = a$. Taking into account that $y = \mathrm{tr}\, L = a + d$ and $D = \det L = ad - bc$, we get $d = y - a = y - D_y$,
$D = ad - bc = D_y(y - D_y) + bD_x$. Hence, L takes the form

$$L = \begin{pmatrix} D_y & b \\ -D_x & y - D_x \end{pmatrix},$$

where $b(x, y)$ is a smooth function satisfying $bD_x = D - D_y(y - D_y)$.

Problem. Describe smooth functions $D(x, y)$ such that

$$b = \frac{D - D_y(y - D_y)}{D_x}$$

is smooth (here we assume that $D_x(0, 0) = 0$).

Examples

$$b = \frac{D - D_y(y - D_y)}{D_x}$$

- Let $D = D(y)$, i.e., $D_x \equiv 0$. Then $D - D_y(y - D_y) = 0$. Differentiating w.r.t. y we get a simpler equation

$$D_{yy}(y - 2D_y) = 0,$$

that shows that $D = \frac{1}{4}y^2 + \gamma$ or $D = \alpha y + \beta$. Substitution into $D - D_y(y - D_y) = 0$ gives $\gamma = 0$ и $\beta = -\alpha^2$. Hence, we have two possibilities

$$D = \frac{1}{4}y^2 \qquad \text{and} \qquad D = \alpha y - \alpha^2,$$

leading to Nijenhuis operators of the form

$$L = \begin{pmatrix} \frac{y}{2} & b(x,y) \\ 0 & \frac{y}{2} \end{pmatrix} \qquad \text{and} \qquad L = \begin{pmatrix} \alpha & b(x,y) \\ 0 & y - \alpha \end{pmatrix}$$

Examples

$$b = \frac{D - D_y(y - D_y)}{D_x}$$

- Let $D = D(x)$, i.e., $D_y \equiv 0$. Then $D - D_y(y - D_y) = D$ and we come to the relation $b(x, y) = \frac{D}{D_x}$.
  If $D_x(0, 0) \neq 0$, then we get a non-singular point. If $D_x(0, 0) = 0$, then obviously $D(0, 0) = 0$. Basically, this is the only restriction if we assume that D is real analytic, i.e., can be represented by a power series

  $$D = a_1 x + a_2 x^2 + a_3 x^3 + \dots$$

  This lead us to the Nijenhuis operator

  $$L = \begin{pmatrix} 0 & \frac{D}{D_x} \\ -D_x & y \end{pmatrix}$$

  Since we can always change the variables x in such a way that $D = \pm x^n$, we come to the following series of Nijenhuis operators

  $$L = \begin{pmatrix} 0 & \frac{1}{n} x \\ \mp n x^{n-1} & y \end{pmatrix}$$

Remark on the eigenvalues

The eigenvalues of a Nijenhuis operator are not necessarily smooth

$$\begin{pmatrix} x & 1 \\ y & 0 \end{pmatrix}$$

The eigenvalues are $\lambda_{1,2} = \dfrac{x \pm \sqrt{x^2 + 4y}}{2}$.

Notice that the eigenvalues become equal on the parabola $y = -\dfrac{1}{4}x^2$.

Moreover,

- for $y > -\dfrac{1}{4}x^2$ the eigenvalues are real and
- for $y < -\dfrac{1}{4}x^2$ the eigenvalues are complex conjugate.

Two real distinct eigenvalues (local setting)

Assume that the eigenvalues of a Nijenhuis operator L are real and distinct. Then they are smooth functions $\lambda_1(x,y)$ and $\lambda_2(x,y)$. Consider the corresponding eigenvectors (vector fields) $\xi_1(x,y)$ and $\xi_2(x,y)$. Without loss of generality we may assume that these vector fields are smooth.

Choose a coordinate system x, y whose coordinate lines coincide with the integral curves of ξ_1 and ξ_2. In other words, ∂_x and ∂_y are eigenvectors too. Then L takes the diagonal form $L = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$.

What we can say about λ_1 and λ_2?

Again we use formula (1): $\mathrm{d}\det L = \mathrm{d}\,\mathrm{tr}\, L \cdot \mathrm{adj}\, L$, i.e.,

$$\big((\lambda_1\lambda_2)_x, (\lambda_1\lambda_2)_y\big) = \big((\lambda_1+\lambda_2)_x, (\lambda_1+\lambda_2)_y\big) \begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_1 \end{pmatrix},$$

which gives two conditions $(\lambda_1 - \lambda_2)(\lambda_2)_x = 0$ and $(\lambda_1 - \lambda_2)(\lambda_1)_y = 0$. Hence, if the eigenvalues are distinct:

$$L = \begin{pmatrix} \lambda_1(x) & 0 \\ 0 & \lambda_2(y) \end{pmatrix}.$$

Complex conjugate eigenvalues (local setting)

Let L have two complex conjugate eigenvalues $\lambda_{1,2} = u \pm iv$. In this case, on M^2 there is a canonical complex structure, i.e., operator J such $J^2 = -\mathrm{Id}$ and $JL = LJ$. These conditions defines J up to sign. The formula for J can be easily written in terms of the eigenvalues $u \pm i\,v$:

$$J = \frac{1}{v}(L - u \cdot \mathrm{Id}) \tag{6}$$

(Exercise: check that $J^2 = -\mathrm{Id}$). Since $\mathcal{N}_J = 0$, we can introduce coordinates x, y such that $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and therefore $L = \begin{pmatrix} u & -v \\ v & u \end{pmatrix}$.

What can we say about functions $u(x, y)$ and $v(x, y)$?

Again we use formula (1): $\mathrm{d}\det L = \mathrm{d}\,\mathrm{tr}\, L \cdot \mathrm{adj}\, L$, i.e.,

$$((u^2 + v^2)_x, (u^2 + v^2)_y) = (2u_x, 2u_y) \begin{pmatrix} u & v \\ -v & u \end{pmatrix}$$

giving 2 conditions: $2vv_x = -2vu_y$ and $2vv_y = 2vu_x$. Since $v \neq 0$, we get

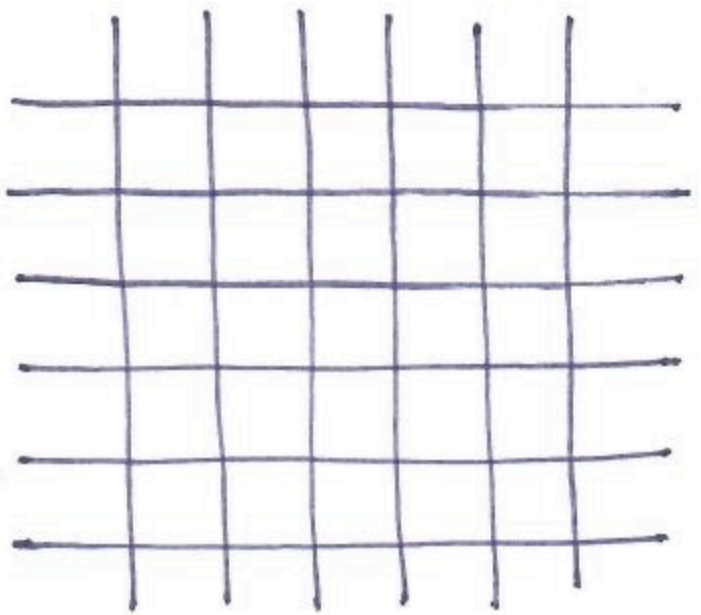
$$u_x = v_y \quad \text{and} \quad u_y = -v_x,$$

meaning that $u(x, y) + iv(x, y)$ is a holomorphic function in $z = x + iy$.

"Portraits" of Nijenhuis operators

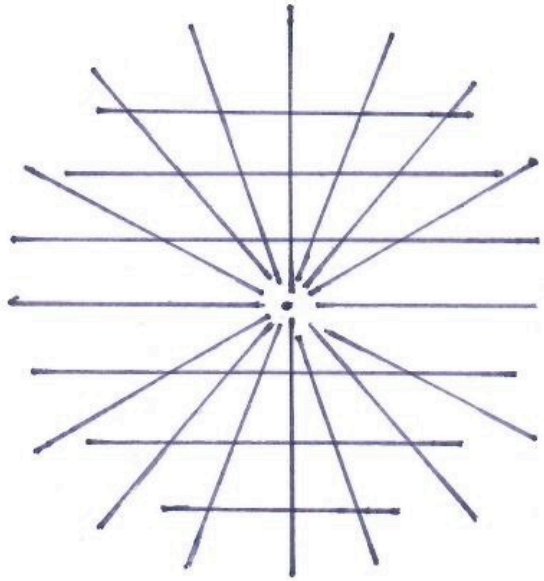
Assume that L admits two distinct real eigenvalues λ_1, λ_2 almost everywhere. Consider the integral curves of the corresponding eigenvectors (vector fields) ξ_1 and ξ_2 (at singular points these vector fields might not be defined, but we can draw these curves almost everywhere). Let us see what they look like in examples.

$$L = \begin{pmatrix} \lambda_1(x) & 0 \\ 0 & \lambda_2(y) \end{pmatrix} \quad \mapsto$$

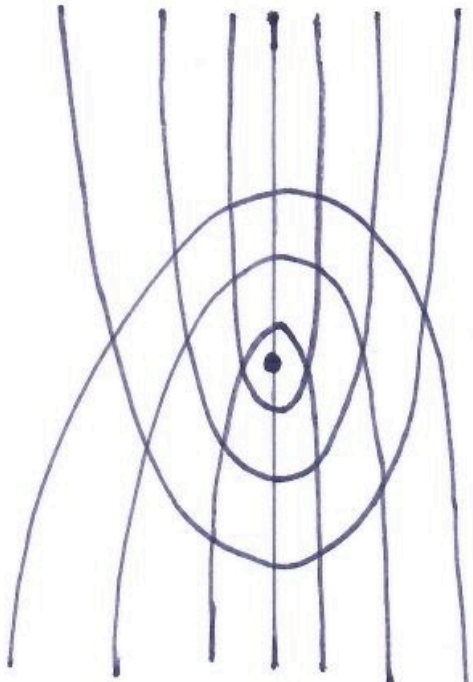


“Portraits” of Nijenhuis operators (examples)

$$L = \begin{pmatrix} 0 & x \\ 0 & y \end{pmatrix} \quad \mapsto$$



$$L = \begin{pmatrix} 0 & x \\ x & 2y \end{pmatrix} \quad \mapsto$$



Construction (of singularities with complicated portraits)

Let $w = u + iv$ be a complex coordinate and $z = z(w) = z(u, v)$ be a holomorphic function, with $z(0) = z'(0) = 0$. Consider the (unique!) Nijenhuis operator L whose eigenvalues are

$$\lambda_1(u, v) = \big(\operatorname{Re} z(u, v)\big)^2 \quad \text{and} \quad \lambda_2(u, v) = -\big(\operatorname{Im} z(u, v)\big)^2.$$

Proposition

L is a real analytic Nijenhuis operator (in a neighbourhood of the origin).

To check this condition, we first consider the operator L in the coordinate system x, y, where $z = (x, y)$. In this coordinate system the eigenvalues are $\lambda_1 = x^2$ and $\lambda_2 = -y^2$ so that the operator takes the form $\begin{pmatrix} x^2 & 0 \\ 0 & -y^2 \end{pmatrix}$, therefore, in coordinates (u, v) we will have:

$$L = \frac{1}{|z'|^2} \begin{pmatrix} y_v & -x_v \\ -y_u & x_u \end{pmatrix} \begin{pmatrix} x^2 & 0 \\ 0 & -y^2 \end{pmatrix} \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix}$$

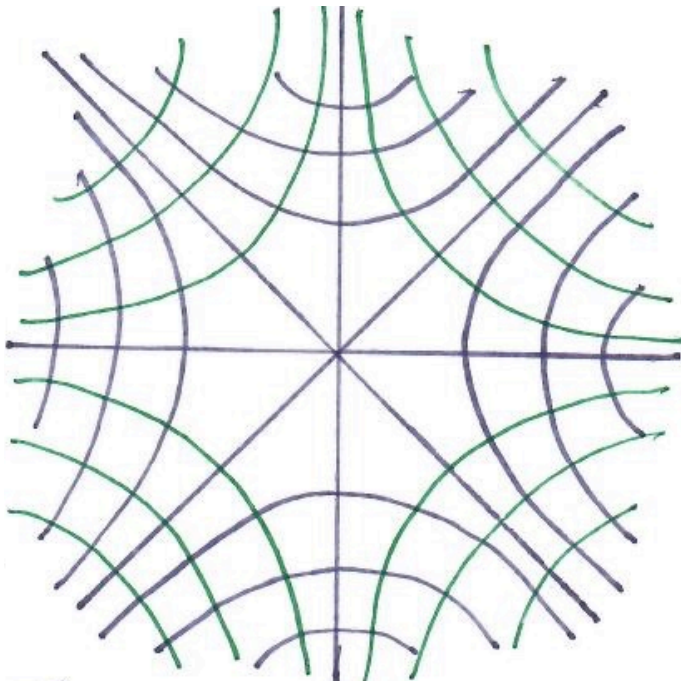
We need to show that L is smooth at the origin.

Example

$$z(w) = w^2, \quad \text{that is } x = u^2 - v^2, y = 2uv, \; z' = 2z,$$

$$L = \frac{1}{u^2+v^2} \begin{pmatrix} u & -v \\ v & u \end{pmatrix} \begin{pmatrix} x^2 & 0 \\ 0 & -y^2 \end{pmatrix} \begin{pmatrix} u & v \\ -v & u \end{pmatrix} =$$

$$\frac{1}{u^2+v^2} \begin{pmatrix} u^2x^2 - v^2y^2 & uv(x^2+y^2) \\ uv(x^2+y^2) & v^2x^2 - u^2y^2 \end{pmatrix} = \begin{pmatrix} u^4 - 3u^2v^2 & uv(u^2+v^2) \\ uv(u^2+v^2) & v^4 - 3v^2u^2 \end{pmatrix}$$



Computation in the general case:

$$\frac{1}{|z'|^2}\begin{pmatrix} y_v & -x_v \\ -y_u & x_u \end{pmatrix}\begin{pmatrix} x^2 & 0 \\ 0 & -y^2 \end{pmatrix}\begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix}$$

We use $x_u = y_v = a$, $x_v = -y_u = b$

$$\frac{1}{a^2+b^2}\begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} x^2 & 0 \\ 0 & -y^2 \end{pmatrix}\begin{pmatrix} a & b \\ -b & a \end{pmatrix} = \begin{pmatrix} \frac{x^2a^2 - y^2b^2}{a^2+b^2} & ab\frac{x^2+y^2}{a^2+b^2} \\ ab\frac{x^2+y^2}{a^2+b^2} & \frac{b^2x^2 - a^2y^2}{a^2+b^2} \end{pmatrix}$$

Why are these functions real analytic?

We use the following fact from complex analysis: if f is holomorphic and $f(0) = 0$, then $\frac{f}{f'}$ is holomorphic. In particular, $\frac{|f|^2}{|f'|^2}$ is real analytic.

Hence, in our case $\frac{|z|^2}{|z'|^2} = \frac{x^2+y^2}{a^2+b^2}$ is real analytic. It remain to notice that

$$\frac{x^2a^2 - y^2b^2}{a^2+b^2} = \frac{x^2a^2 + x^2b^2 - (x^2+y^2)b^2}{a^2+b^2} = x^2 - b^2\frac{x^2+y^2}{a^2+b^2}.$$

Similar for the L_2^2 component.

Jordan block in dimension 2

We know already that a typical Nijenhuis Jordan block in dimension two takes the form $\begin{pmatrix} g(y) & f(x,y) \\ 0 & g(y) \end{pmatrix}$. Notice that the eigenvalue λ of L in this case is a smooth function as $\lambda = \frac{1}{2}\operatorname{tr} L$.

Question. Let M^2 be a closed orientable surface and $f : M^2 \to \mathbb{R}$ be a smooth function. Can one construct a Nijenhuis operator L that is similar to a Jordan block with eigenvalue f everywhere except, perhaps, for critical points of f)?
The answer is YES. Consider

$$L = f \cdot \mathrm{Id} + X_f \otimes \mathrm{d}\, f$$

where X_f denotes a Hamiltonian vector field for f w.r.t. some symplectic form ω on M^2.

In local coordinates (x, y): $\omega = h(x,y)\mathrm{d}\, x \wedge \mathrm{d}\, y$, $X_f = h^{-1}\begin{pmatrix} -f_y \\ f_x \end{pmatrix}$. Then

$$L = f \cdot \mathrm{Id} + X_f \otimes \mathrm{d}\, f = \begin{pmatrix} f - \frac{f_x f_y}{h} & -\frac{f_y^2}{h} \\ \frac{f_x^2}{h} & f + \frac{f_x f_y}{h} \end{pmatrix}, \quad \det L = f^2, \ \operatorname{tr} L = 2f. \quad (7)$$

Verification of $\mathrm{d}\det L = \mathrm{d}\operatorname{tr} L \cdot \operatorname{adj}(L)$ is straightforward.

Exercises

Exercise

Show that the operator defined by (7)

- is similar to the Jordan block with eigenvalue f,
- is a Nijenhuis operator.

Exercise

Let L be an operator with two complex conjugate eigenvalues $\lambda_{1,2} = u \pm iv$. Show that the operator J defined by (6) satisfies the following properties

- $J^2 = -\mathrm{Id}$;
- $LJ = JL$.

Prove that there are only two operators with these properties, J and $-J$.

Exercise

Compute the eigenvalues of the operator $L = \begin{pmatrix} 0 & x \\ x & 2y \end{pmatrix}$ and check that the portrait of this operator is as shown on Slide 16.

Exercise

Under which conditions on $a, b \in \mathbb{R}$, the Nijenhuis operator L with $\operatorname{tr} L = (ax + b)y$ and $\det L = y^2$ is smooth at the origin?