Nijenhuis Geometry

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Lecture 5:
Normal forms of Nijenhuis operator
and Haantjes torsion

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Facts, we will need

- Straightening theorem for vector fields: consider a vector field ξ around point p such that $\xi(\mathrm{p}) \neq 0$. Then there exists a coordinate system $x^1, \ldots, x^n$, centered at p, such that $\xi = \partial_{x^1}$.
- Consider vector fields $\xi_1, \ldots, \xi_k$, defined in a neighbourhood of p, such that $\xi_1(p), \ldots, \xi_k(p)$ are linearly independent and $[\xi_i, \xi_j] = 0$. Then there exits a coordinate system $x^1, \ldots, x^k, y^1, \ldots, y^{n-k}$, such that $\xi_i = \partial_{x^i}$

Nijenhuis theorem

Theorem (Nijenhuis, 1951)

Consider a Nijenhuis operator L with real pairwise distinct eigenvalues at each point. Then there exists a coordinate system, in which the operator L can be brought to the form

$$L = \begin{pmatrix} \lambda_1(x^1) & 0 & \dots & 0 \\ 0 & \lambda_2(x^2) & \dots & 0 \\ & & \ddots & \\ 0 & 0 & \dots & \lambda_n(x^n) \end{pmatrix}.$$

Proof: The characteristic polynomial of L at a point p is factorised as

$$\chi_{L(\mathrm{p})}(t) = (t - \lambda_1(\mathrm{p}))\dots(t - \lambda_n(\mathrm{p})).$$

Applying the Splitting Theorem we get the result.

The simplest normal form

The coordinate system, in which the operator is diagonal, is not unique. Consider a coordinate change $x(y)$. The operator is transformed as

$$L \to \left(\frac{\partial x}{\partial y}\right)^{-1} L \left(\frac{\partial x}{\partial y}\right).$$

For operator to stay diagonal we have that x^i must depend only on y^i. Thus the operator in new coordinates takes form

$$L = \begin{pmatrix} \lambda_1(x^1(y^1)) & 0 & \dots & 0 \\ 0 & \lambda_2(x^2(y^2)) & \dots & 0 \\ & & \ddots & \\ 0 & 0 & \dots & \lambda_n(x^n(y^n)) \end{pmatrix}$$

The simplest normal form

We arrive to the following problem: given function $f(x)$ of single variable, what is the simplest form, you can bring it to with substitution $x(y)$ for $x' \neq 0$?

Proposition

Consider a real analytic non-constant function $f(x)$, such that $f(0) = c$. Then there exists a change of variable $x(y)$, such that the function is brought to exactly one of these forms

1. $f(x) = c + y^k$ for odd k
2. $f(x) = c + y^k$ for even k
3. $f(x) = c - y^k$ for even k

Proof: Denote the order of zero $f(x) - c$ by $k \geq 1$. Then we have $f(x) = c + x^k(a + \dots)$ for $a \neq 0$. Then $y = (|f - c|)^{\frac{1}{k}}$. This function is analytic in a neighbourhood of the zero, thus, yields the appropriate coordinate change. We see that for odd k such coordinate change is unique, while for even k we have two coordinate changes $y = \pm(|f - c|)^{\frac{1}{k}}$.

The simplest normal form

Thus, we have proven the following theorem

Theorem

Consider a real analytic Nijenhuis operator L with real pairwise distinct non-constant eigenvalues. Then there exists a finite number of coordinate systems, centered at point p, in which the operator L is in the form

$$L = \begin{pmatrix} \lambda_1(x^1) & 0 & \dots & 0 \\ 0 & \lambda_2(x^2) & \dots & 0 \\ & & \ddots & \\ 0 & 0 & \dots & \lambda_n(x^n) \end{pmatrix}$$

where $\lambda_i = c_i + \delta_i (x^i)^{k_i}$. Here c_i are real constants, k_i are natural numbers and $\delta_i = 1$ for odd k_i and ± 1 for even k_i. These constants uniquely define normal form. The number of corresponding coordinate systems is 2^s, where s is the number of even k_i's. In particular, if all k_i's are odd, then such a coordinate system is unique.

The simplest normal form

Two important comments:

- If eigenvalue λ_i is constant, then every coordinate change in the form $x^i(y^i)$ preserves the normal form. In particular, for constant operator

$$L = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ & & \ddots & \\ 0 & 0 & \dots & \lambda_n \end{pmatrix}$$

  every coordinate change $x^i(y^i)$ preserves the normal form
- If we do not fix point p, then shifts allow us to change c_i in the normal form in the theorem

Diagonalization problem

The original problem, which was studied by Nijenhuis, was the following: given an operator field with distinct real eigenvalues, when does there exist a coordinate system, in which the operator is diagonal?

It was solved earlier by his teacher Schouten in a following manner: given a pointwise diagonal operator L one can construct a metric g, such that $g(L\xi, \eta) = g(\xi, L\eta) = h(\xi, \eta)$. Denote the eigenvector fields as $\xi_1, \dots, \xi_n$.

Then the existence condition for coordinate system can be written as

$$\nabla_\alpha h_{\beta\gamma} \xi_i^\alpha \xi_j^\beta \xi_k^\gamma = 0, \quad i \neq j, j \neq k, i \neq k.$$

Here ∇ is a Levi-Civita connection of g. Nijenhuis was able to get rid of metric and express the condition only using $\mathcal{N}_L$ and eigenvectors.

Example of non-diagonalizable operator

Consider the operator field:

$$L = \begin{pmatrix} 1 & 0 & 2y \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

on 3-dimensional affine space with coordinates x, y, z.

The eigenvalues are constant and $1, 2, 3$. At the same time the corresponding vector fields are $\partial_x, \partial_y, \partial_z + y\partial_x$.

Example of non-diagonalizable operator

We need new coordinates, but not all the coordinate changes are allowed. The only coordinate change we can perform is the change $\xi_i \to f_i\xi_i$ for f_i being some function. In our case this yields the following system of PDEs on three non-zero functions f, g, h:

$$[f\partial_x, g\partial_y] = 0, \quad [f\partial_x, h\partial_z + hy\partial_x] = 0, \quad [g\partial_y, h\partial_z + hy\partial_x] = 0.$$

This can be written as

$$\begin{aligned} &gf_y\partial_x - fg_x\partial_y = 0, \\ &\Big(hf_z + y(hf_x - fh_x)\Big)\partial_x - fh_x\partial_z = 0, \\ &(-gh_y y - gh)\partial_x + h(yg_x + g_z)\partial_y - gh_y\partial_z = 0 \end{aligned}$$

We see that the last equation yields $h_y = 0$ and $g(h_y y + h) = 0$. Thus, $h \equiv 0$ and there is no solution to the system. Thus, the operator cannot be brought to the diagonal form.

Special Frobenius Theorem

Theorem (Frobenius 3)

Consider a collection of vector fields $\xi_1, \ldots, \xi_k$ *on an* $n-$*dimensional manifold. The following two statements are equivalent:*

1. *There exists a collection of non-zero functions* $f_1, \ldots, f_k$ *such that*

$$[f_i\xi_i, f_j\xi_j] = 0$$

2. *For all pairs* i, j *we have* $[\xi_i, \xi_j] \in \mathcal{D}_{ij}$*, where* $\mathcal{D}_{ij} = \mathrm{span}\langle \xi_i, \xi_j \rangle$

This theorem provides the necessary and sufficient condition for diagonalization to exists. In our example vector fields $[\partial_y, \partial_z + y\partial_x] = -\partial_x$. Thus, the commutator of these vector fields did not belong to the distribution, spanned by $\langle \partial_y, \partial_z + y\partial_x \rangle$

Proof of Frobenius 3: step 1

Let us show that the first statement implies the second.

Indeed, denote $\eta_i = f_i \xi_i$ and we have that $[\eta_i, \eta_j] = 0$. At the same time

$$[\xi_i, \xi_j] = \left[\frac{1}{f_i}\eta_i, \frac{1}{f_j}\eta_j\right] =$$
$$= -\frac{1}{f_j}\mathcal{L}_{\eta_j}\left(\frac{1}{f_i}\right)\eta_i + \frac{1}{f_i}\mathcal{L}_{\eta_i}\left(\frac{1}{f_j}\right)\eta_j = -f_i\mathcal{L}_{\xi_j}\left(\frac{1}{f_i}\right)\xi_i + f_j\mathcal{L}_{\xi_i}\left(\frac{1}{f_j}\right)\xi_j.$$

Thus, the commutator lies in $\mathcal{D}_{ij}$.

Observation: We see (as expected), that the second condition of the Theorem does not change if we multiply vector fields by functions

Proof of Frobenius 3: step 2

Now let us show that the second statement implies the first. The proof is by induction.

For $k = 2$ we have a single involutive two-dimensional distribution on $n-$dimensional manifold. In local coordinates we have $[\xi_1, \xi_2] = a\xi_1 + b\xi_2$ for some functions a, b. This yields

$$[f_1\xi_1, f_2\xi_2] = f_1\mathcal{L}_{\xi_1}(f_2)\xi_2 - f_2\mathcal{L}_{\xi_2}(f_1)\xi_1 + f_1 f_2[\xi_1, \xi_2] =$$
$$= f_2\Big(f_1 a - \mathcal{L}_{\xi_2} f_1\Big)\xi_1 + f_1\Big(\mathcal{L}_{\xi_1} f_2 + f_2 b\Big)\xi_2$$

We get the system, that **looks like** a system of two PDEs

$$\begin{cases} \mathcal{L}_{\xi_2} f_1 - f_1 a = 0, \\ \mathcal{L}_{\xi_1} f_2 + f_2 b = 0. \end{cases}$$

But we see right on that two equations are independent.

Proof of Frobenius 3: step 2

Take the first equation and consider vector field ξ_2. Applying the straightening theorem for vector fields, we get that the equation in some coordinates $x^1, y^1, \ldots, y^{n-1}$ takes form

$$\frac{\partial f_1}{\partial x^1} - f_1 a = 0 \to \frac{1}{f_1}\frac{\partial f_1}{\partial x^1} = a \to f_1 = C\exp(\int a(x^1, y)\,\mathrm{d}\,x^1).$$

Here $\int a(x^1, y)\,\mathrm{d}\,y^1$ stands for antiderivative of a in variable x^1.

Choosing constant C we get the condition $f_1(0,0) = 1$. Same goes for f_2 as two systems can be solved independently.

Proof of Frobenius 3: step 3

Now assume that the statement is proved for k. Due to earlier observation we may assume that $\xi_1, \dots, \xi_k$ commute. Applying Frobenius theorem, we get that in appropriate coordinate system $x^1, \dots, x^k, y^1, \dots, y^{n-k}$ the given vector fields are:

$$\xi_1 = \partial_{x^1}, \quad \dots, \quad \xi_k = \partial_{x^k}, \quad \xi_{k+1} = \xi.$$

The Jacobi condition for $\xi, \partial_{x^i}, \partial_{x^j}$ yields

$$0 = [\partial_{x^j}, [\xi, \partial_{x^i}]] - [\partial_{x^i}, [\xi, \partial_{x^j}]] = [\partial_{x^j}, a_i\xi + b_i\partial_{x^i}] - [\partial_{x^i}, a_j\xi + a_j\partial_{x^j}] =$$
$$= \frac{\partial a_i}{\partial x^j}\xi + a_i[\xi, \partial_{x^j}] + \frac{\partial b_i}{\partial x^j}\partial_{x^i} - \frac{\partial a_j}{\partial x^i}\xi - a_j[\xi, \partial_{x^i}] - \frac{\partial b_j}{\partial x^i}\partial_{x^j} =$$
$$= \left(\frac{\partial a_i}{\partial x^j} - \frac{\partial a_j}{\partial x^i}\right)\xi + \left(a_i b_j - \frac{\partial b_j}{\partial x^i}\right)\partial_{x^j} - \left(a_j b_i - \frac{\partial b_i}{\partial x^j}\right)\partial_{x^i}$$

Vanishing of blue part implies that there exists function F such that $\frac{\partial F}{\partial x^i} = a_i$.

Proof of Frobenius 3: step 4

For F we note that

$$[\exp(-F)\xi, \partial_{x^j}] = -a_i \exp(-F)\xi + \exp(-F)[\xi, \partial_{x^i}] = -\exp(-F) b_i \partial_{x^j}.$$

Multiplying ξ by the $\exp(-F)$ and leaving the same notations, we get that in our collection of vector fields $\partial_{x^1}, \ldots, \partial_{x^k}, \xi$ first k vector fields commute and

$$[\xi, \partial_{x^i}] = b_i \partial_{x^i}.$$

Write ξ in coordinates

$$\xi = \xi^1 \partial_{x^1} + \cdots + \xi^k \partial_{x^k} + \bar{\xi}^1 \partial_{y^1} + \cdots + \bar{\xi}^{n-k} \partial_{y^{n-k}}.$$

The aforementioned condition implies, that the coefficients ξ^i depend only on x^i and $y^1, \ldots, y^{n-k}$, while $\bar{\xi}^j$ depend only on $y^1, \ldots, y^{n-k}$. In particular, $b_i = \frac{\partial \xi^i}{\partial x^i}$.

Proof of Frobenius 3: step 4

Now we search for functions g_i, $i = 1, \dots, k$, such that each g_i depends on x^i and $y^1, \dots, y^{n-k}$ and

$$[\xi, g_i \partial_{x^i}] = (g_i b_i - \xi(g_i))\partial_{x^i} = 0.$$

One can see that we get a system similar to the two-dimensional case: k equations on k functions, all are independent of each other.
Take the first equation that is equation on g_1. In coordinate form it is

$$g_1 \frac{\partial \xi^1}{\partial x^1} - \xi^1 \frac{\partial g_1}{\partial x^1} - \sum_{j=1}^{n-k} \bar{\xi}^j \frac{\partial g_1}{\partial y^j} = 0.$$

This equation can be understood as an equation on "reduced" $n-k+1$-dimensional space with coordinates $x^1, y^1, \dots, y^{n-k}$ for vector field

$$\xi_{\mathrm{red}} = \xi^1 \partial_{x^1} + \bar{\xi}^\alpha \partial_{y^\alpha}.$$

Proof of Frobenius 3: step 5

Now we observe that

$$[g_i \partial_{x^i}, g_j \partial_{x^j}] = 0$$

and

$$[\xi, g_i \partial_{x^i}] = 0.$$

Thus, the first condition of the theorem holds.

The theorem is proved.

Haantjes torsion

The condition found by Schouten was rewritten by Nijenhuis. Later Haantjes in his work in 1955 gave the tensorial conditon for the existence of the corresponding coordinate system.

For operator field L consider Haantjes torsion

$$\mathcal{H}_L(\xi, \eta) = L\mathcal{N}_L(L\xi, \eta) + L\mathcal{N}_L(\xi, L\eta) - \mathcal{N}_L(L\xi, L\eta) - L^2\mathcal{N}_L(\xi, \eta).$$

Here $\mathcal{N}_L$ is Nijenhuis torsion. Note that this condition is of order one in derivatives of L but of order three in components of L. The coordinate formula for Haantjes torsion is

$$\begin{aligned}(\mathcal{H}_L)^k_{ij} = L^k_\alpha (\mathcal{N}_L)^\alpha_{\beta j} L^\beta_i + L^k_\alpha (\mathcal{N}_L)^\alpha_{i\beta} L^\beta_j + \\ - L^k_\alpha L^\alpha_\beta (\mathcal{N}_L)^\beta_{ij} - (\mathcal{N}_L)^k_{\alpha\beta} L^\alpha_i L^\beta_j\end{aligned} \tag{1}$$

Haantjes torsion

If L is Nijenhuis operator, then fL for some function f is not Nijenhuis. In particular,

$$\begin{aligned}\mathcal{N}_{fL}(\xi,\eta) &= fL[fL\xi,\eta] + fL[\xi, fL\eta] - f^2L^2[\xi,\eta] - [fL\xi, fL\eta] = \\ &= -f\mathcal{L}_\eta(f)L^2\xi + f^2L[L\xi,\eta] + f\mathcal{L}_\xi(f)L^2\eta + f^2L[\xi, L\eta] - f^2L[\xi,\eta] + \\ &+ f\mathcal{L}_{L\eta}(f)L\xi - f\mathcal{L}_{L\xi}(f)L\eta - f^2[L\xi, L\eta] = \\ &= f^2\mathcal{N}_L(\xi,\eta) - f\Big(\mathcal{L}_\eta(f)L^2 - \mathcal{L}_{L\eta}(f)L\Big)\xi + \\ &+ f\Big(\mathcal{L}_\xi(f)L^2 - \mathcal{L}_{L\xi}(f)L\Big)\eta\end{aligned}$$

Theorem (Scaling property of Haantjes torsion)

The Haantjes torsions of L and fL are related as $\mathcal{H}_{fL}(\xi,\eta) = f^4\mathcal{H}_L(\xi,\eta)$

Proof of scaling property of Haantjes torsion

The direct calculation of the Haantjes torsion yields

$$
\begin{aligned}
&\mathcal{H}_{fL}(\xi,\eta) = \\
&= f^2 L\mathcal{N}_{fL}(L\xi,\eta) + f^2 L\mathcal{N}_{fL}(\xi,L\eta) - f^2\mathcal{N}_{fL}(L\xi,L\eta) - f^2L^2\mathcal{N}_{fL}(\xi,\eta) = \\
&= f^4\mathcal{H}_L(\xi,\eta) + \\
&- f^3\Big(\mathcal{L}_\eta(f)L^4 - \mathcal{L}_{L\eta}(f)L^3\Big)\xi + f^3\Big(\mathcal{L}_{L\xi}(f)L^3 - \mathcal{L}_{L^2\xi}(f)L^2\Big)\eta + \\
&- f^3\Big(\mathcal{L}_{L\eta}(f)L^3 - \mathcal{L}_{L^2\eta}(f)L^2\Big)\xi + f^3\Big(\mathcal{L}_\xi(f)L^4 - \mathcal{L}_{L\xi}(f)L^3\Big)\eta + \\
&+ f^3\Big(\mathcal{L}_{L\eta}(f)L^3 - \mathcal{L}_{L^2\eta}(f)L^2\Big)\xi - f^3\Big(\mathcal{L}_{L\xi}(f)L^3 - \mathcal{L}_{L^2\xi}(f)L^2\Big)\eta + \\
&+ f^3\Big(\mathcal{L}_\eta(f)L^4 - \mathcal{L}_{L\eta}(f)L^3\Big)\xi - f^3\Big(\mathcal{L}_\xi(f)L^4 - \mathcal{L}_{L\xi}(f)L^3\Big)\eta = \\
&= f^4\mathcal{H}_L(\xi,\eta)
\end{aligned}
$$

The theorem is proved.

Haantjes theorem

Theorem (Haantjes, 1955)

Consider an operator field L with real pairwise distinct eigenvalues at each point. Then two conditions are equivalent

1. *There exists a coordinate system, in which L is diagonal*

$$L = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ & & \ddots & \\ 0 & 0 & \dots & \lambda_n \end{pmatrix}.$$

   Here each function λ_i depends on all the variables.

2. *Haantjes torsion of L vanishes.*

Proof of Haantjes theorem: step 1

Assume that L is diagonal in coordinates $x^1, \dots, x^n$. In this case the Nijenhuis torsion takes the form

$$\begin{aligned}\mathcal{N}_L(\partial_{x^i}, \partial_{x^j}) &= L[\lambda_i \partial_{x^i}, \partial_{x^j}] + L[\partial_{x^i}, \lambda_j \partial_{x^j}] - [\lambda_i \partial_{x^i}, \lambda_j \partial_{x^j}] = \\ &= -\frac{\partial \lambda_i}{\partial x^j} \lambda_i \partial_{x^i} + \frac{\partial \lambda_j}{\partial x^i} \lambda_j \partial_{x^j} + \lambda_j \frac{\partial \lambda_i}{\partial x^j} \partial_{x^i} - \lambda_i \frac{\partial \lambda_j}{\partial x^i} \partial_{x^j} = \\ &= (\lambda_j - \lambda_i) \frac{\partial \lambda_i}{\partial x^j} \partial_{x^i} + (\lambda_j - \lambda_i) \frac{\partial \lambda_j}{\partial x^i} \partial_{x^j}.\end{aligned}$$

Proof of Haantjes theorem: step 1

The Haantjes torsion takes form

$$\begin{aligned}\mathcal{H}_L(\partial_{x^i},\partial_{x^j}) &= L\mathcal{N}(L\partial_{x^i},\partial_{x^j}) + L\mathcal{N}_L(\partial_{x^i},L\partial_{x^j}) - L^2\mathcal{N}_L(\partial_{x^i},\partial_{x^j})-\\ &-\mathcal{N}_L(L\partial_{x^i},L\partial_{x^j}) =\\ &= (\lambda_j-\lambda_i)\lambda_i^2\frac{\partial\lambda_i}{\partial x^j}\partial_{x^i} + (\lambda_j-\lambda_i)\lambda_i\lambda_j\frac{\partial\lambda_j}{\partial x^i}\partial_{x^j}+\\ &+(\lambda_j-\lambda_i)\lambda_i\lambda_j\frac{\partial\lambda_i}{\partial x^j}\partial_{x^i} + (\lambda_j-\lambda_i)\lambda_j^2\frac{\partial\lambda_j}{\partial x^i}\partial_{x^j}-\\ &-(\lambda_j-\lambda_i)\lambda_i^2\frac{\partial\lambda_i}{\partial x^j}\partial_{x^i} - (\lambda_j-\lambda_i)\lambda_j^2\frac{\partial\lambda_j}{\partial x^i}\partial_{x^j}-\\ &-(\lambda_j-\lambda_i)\lambda_i\lambda_j\frac{\partial\lambda_i}{\partial x^j}\partial_{x^i} - (\lambda_j-\lambda_i)\lambda_i\lambda_j\frac{\partial\lambda_j}{\partial x^i}\partial_{x^j} =\\ &= 0.\end{aligned}$$

Proof of Haantjes theorem: step 2

Now assume that Haantjes torsion vanishes and $\xi_1, \dots, \xi_n$ are eigenvector fields with eigenvalues $\lambda_1, \dots, \lambda_n$. For Nijenhuis torsion we have

$$\begin{aligned}
&\mathcal{N}_L(\xi_i, \xi_j) = L[L\xi_i, \xi_j] + L[\xi_i, L\xi_j] - [L\xi_i, L\xi_j] - L^2[\xi_i, \xi_j] = \\
&= -\mathcal{L}_{\xi_j}(\lambda_i)\lambda_i\xi_i + \lambda_i L[\xi_i, \xi_j] + \mathcal{L}_{\xi_j}(\lambda_i)\lambda_j\xi_j + \lambda_j L[\xi_i, \xi_j] + \\
&+ \mathcal{L}_{\xi_j}(\lambda_i)\lambda_j\xi_i - \mathcal{L}_{\xi_i}(\lambda_j)\lambda_i\xi_j + \lambda_i\lambda_j[\xi_i, \xi_j] - L^2[\xi_i, \xi_j] = \\
&= (\lambda_j - \lambda_i)\mathcal{L}_{\xi_j}(\lambda_i)\xi_i + (\lambda_j - \lambda_i)\mathcal{L}_{\xi_i}(\lambda_j)\xi_j - \\
&- \Big(L - \lambda_i \operatorname{Id}\Big)\Big(L - \lambda_j \operatorname{Id}\Big)[\xi_i, \xi_j]
\end{aligned}$$

Proof of Haantjes theorem: step 2

Substituting into Haantjes torsion we get

$$
\begin{aligned}
&\mathcal{H}_L(\xi_i,\xi_j) = \\
&= (\lambda_j-\lambda_i)\mathcal{L}_{\xi_j}(\lambda_i)\lambda_i^2\xi_i + (\lambda_j-\lambda_i)\mathcal{L}_{\xi_i}(\lambda_j)\lambda_i\lambda_j\xi_j - \\
&- \lambda_i L\Big(L-\lambda_i \operatorname{Id}\Big)\Big(L-\lambda_j \operatorname{Id}\Big)[\xi_i,\xi_j] + \\
&+ (\lambda_j-\lambda_i)\mathcal{L}_{\xi_j}\lambda_i\lambda_j(\lambda_i)\xi_i + (\lambda_j-\lambda_i)\mathcal{L}_{\xi_i}(\lambda_j)\lambda_j^2\xi_j - \\
&- \lambda_j L\Big(L-\lambda_i \operatorname{Id}\Big)\Big(L-\lambda_j \operatorname{Id}\Big)[\xi_i,\xi_j] - \\
&- (\lambda_j-\lambda_i)\mathcal{L}_{\xi_j}(\lambda_i)\lambda_i\lambda_j\xi_i - (\lambda_j-\lambda_i)\mathcal{L}_{\xi_i}(\lambda_j)\lambda_i\lambda_j\xi_j + \\
&+ \lambda_i\lambda_j\Big(L-\lambda_i \operatorname{Id}\Big)\Big(L-\lambda_j \operatorname{Id}\Big)[\xi_i,\xi_j] - \\
&- (\lambda_j-\lambda_i)\mathcal{L}_{\xi_j}(\lambda_i)\lambda_i^2\xi_i - (\lambda_j-\lambda_i)\mathcal{L}_{\xi_i}(\lambda_j)\lambda_j^2\xi_j + \\
&+ L^2\Big(L-\lambda_i \operatorname{Id}\Big)\Big(L-\lambda_j \operatorname{Id}\Big)[\xi_i,\xi_j] = \\
&= \Big(L-\lambda_i \operatorname{Id}\Big)^2\Big(L-\lambda_j \operatorname{Id}\Big)^2[\xi_i,\xi_j] = 0
\end{aligned}
$$

We get, that $[\xi_i,\xi_j] \in \operatorname{span}\langle \xi_i,\xi_j\rangle$. Theorem is proved.

Exercises for Lecture 5

1. Prove that Haantjes torsion vanishes for all operator fields in dimension two.
2. Prove the Schouten condition for the diagonalization problem.
3. Find a formula for Haantjes torsion that involves only commutators of vector fields.
4. Using scaling property of Haantjes operator, prove that for arbitrary functions f, g the identity holds

$$\mathcal{H}_{fL+g\,\mathrm{Id}}(\xi,\eta) = f^4\mathcal{H}_L(\xi,\eta).$$