Nijenhuis Geometry

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Lecture 6:
Constant normal forms, nilpotent Nijenhuis operators and Thompson theorem

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Spring 2021

Facts, we will need

- Frobenius 2 theorem is reformulated as: for linearly independent and pairwise commuting $\xi_1, \ldots, \xi_k$ there exists a collection of functions $f^1, \ldots, f^k, g^1, \ldots, g^m$, such that $\xi_i(f^j) = \delta_i^j, \xi_i(g^j) = 0$. The condition on functions imply that differentials of $\mathrm{d}\, f^i$ are automatically linearly independent and span$\langle \mathrm{d}\, f^1, \ldots, \mathrm{d}\, f^k \rangle$ and span$\langle \mathrm{d}\, g^1, \ldots, \mathrm{d}\, g^m \rangle$ do not intersect. Thus, to insure that given functions define coordinate system, we need to check that $\mathrm{d}\, g^j$ are linearly independent.

Facts, we will need

- For linearly independent and commuting $\xi_1, \dots, \xi_k$ assume that we have found l functionally independent integrals $g^1, \dots, g^l$. Then we can find the rest $m - l$ integrals to get the coordinate system. To do that pick the coordinate system, adopted to distribution $x^1, \dots, x^k, y^1, \dots, y^m$. By definition in these coordinates $\partial_{x^i} g^j = 0$, that is these functions depend only on $y^1, \dots, y^m$. Consider the matrix

$$\begin{pmatrix} \frac{\partial g^1}{\partial y^1} & \dots & \frac{\partial g^1}{\partial y^m} \\ & \dots & \\ \frac{\partial g^l}{\partial y^1} & \dots & \frac{\partial g^l}{\partial y^m} \end{pmatrix}$$

  The rank of matrix is l, so w.l.o.g. we assume that the first $l \times l$ minor is not zero. Then $\mathrm{d}\, g^1, \dots, \mathrm{d}\, g^l, \mathrm{d}\, y^{l+1}, \dots, \mathrm{d}\, y^m$ are linearly independent. This is a complete collection of the integrals.

Quotient operator and L-invariant distribution

Linear algebra: consider vector space V and subspace W. Given an operator L such that W is invariant under the action of operator, we can define two natural operators L_1 and L_2:

- L_1 is a restriction of L onto W
- L_2 is a quotient operator $L_2 : V/W \to V/W$.

The latter defined as follows: element $\xi + W \in V/W$ is an equivalence class. Pick a representative $\xi + \eta$, where $\eta \in W$. We have that $L_2(\xi + W) = L(\xi + \eta) = L\xi + L\eta$ and $L\eta \in W$. Thus, the result does not depend on the choice of representative of equivalence class.

In basis $\eta_1, \ldots, \eta_k, \xi_1, \ldots, \xi_m$, where η_i is basis in W, while ξ_i can be treated as basis in V/W. In this basis the operator takes the form

$$L = \begin{pmatrix} L_1 & M \\ 0 & L_2 \end{pmatrix}$$

Quotient operator and L-invariant distribution

Consider operator field L and integrable distribution $\mathcal{D}$, invariant under L, that is for arbitrary vector field $\xi \in \mathcal{D}$ the vector field $L\xi \in \mathcal{D}$.

The corresponding foliation we denote as $\mathcal{F}$. Pick coordinate system $x^1, \dots, x^k, y^1, \dots, y^m$, adapted to $\mathcal{D}$. Then, according to the previous slides, operator field takes form

$$L = \begin{pmatrix} L_1(x,y) & M(x,y) \\ 0 & L_2(x,y) \end{pmatrix}$$

The coordinates x^i are coordinates on the foliation $\mathcal{F}$, while $y^1, \dots, y^m$ are coordinates on quotient space $M/\mathcal{F}$. To define quotient operator we need the block L_2 not to depend on x.

Quotient operator and L-invariant distribution

The vector field ξ is said to be $\mathcal{D}-$**preserving**, if for any $\eta \in \mathcal{D}$ we have that $[\xi, \eta] \in \mathcal{D}$. In adapted coordinate system the $\mathcal{D}-$preserving vector field is written as

$$\xi = \xi^1 \partial_{x^1} + \cdots + \xi^k \partial_{x^k} + \bar{\xi}^1 \partial_{y^1} + \cdots + \bar{\xi}^m \partial_{y^m}.$$

where $\bar{\xi}^i$ do not depend on x.

We see that L_2 does not depend on x if and only if for all $D-$preserving ξ the vector field $L\xi$ is also $\mathcal{D}-$preserving. Thus, we have proven the Proposition.

Proposition

Consider $L-$invariant integrable distribution $\mathcal{D}$ with integral foliation $\mathcal{F}$. The quotient operator field on $M/\mathcal{F}$ is well-defined if and only if for every $\mathcal{D}-$preserving vector field ξ the vector field $L\xi$ is also $\mathcal{D}-$preserving.

Quotient Nijenhuis operator

Theorem

Consider Nijenhuis operator L and $L-$invariant integrable distribution $\mathcal{D}$ with integral foliation $\mathcal{F}$. If the quotient operator field on $M/\mathcal{F}$ is correctly defined, then it is Nijenhuis operator.

Proof: Fix the same coordinate system $x^1, \dots, x^k, y^1, \dots, y^m$ as before, adapted to the distribution $\mathcal{D}$ (and foliation $\mathcal{F}$). We have:

$$0 = L^2[\partial_{y^i}, \partial_{y^j}] - L[L\partial_{y^i}, \partial_{y^j}] - L[\partial_{y^i}, L\partial_{y^j}] + [L\partial_{y^i}, L\partial_{y^j}] =$$
$$= -L[M_i^\alpha(x,y)\partial_{x^\alpha} + (L_2)_i^\gamma(y)\partial_{y^\gamma}, \partial_{y^j}] - L[\partial_{y^i}, M_j^\alpha(x,y)\partial_{x^\alpha} + (L_2)_j^\nu(y)\partial_{y^\nu}] +$$
$$+ [M_i^\alpha(x,y)\partial_{x^\alpha} + (L_2)_i^\gamma(y)\partial_{y^\gamma}, M_j^\beta(x,y)\partial_{x^\beta} + (L_2)_j^\nu(y)\partial_{y^\nu}] =$$
$$= \Big((L_2)_j^\nu \frac{\partial (L_2)_i^\gamma}{\partial y^\nu} - (L_2)_i^\nu \frac{\partial (L_2)_j^\gamma}{\partial y^\nu} + (L_2)_i^\nu \frac{\partial (L_2)_j^\gamma}{\partial y^\nu} - (L_2)_j^\nu \frac{\partial (L_2)_i^\gamma}{\partial y^\nu}\Big)\partial_{y^\gamma} + \dots$$

The $\dots$ stand for vector field $\in \operatorname{span}(\partial_{x_1}, \dots, \partial_{x_k})$. Vanishing of the entire Nijenhuis torsion implies vanishing of this condition, which is exactly the Nijenhuis condition for the quotient operator.

Constant normal form

So back to the normal forms.

Question: When operator field L with real eigenvalues can be brought to constant form?

The necessary conditions are:

- Nijenhuis torsion for constant operator vanishes, thus L is Nijenhuis operator
- The eigenvalues of the operator are constant functions
- The number and structure of Jordan blocks do not change

Applying Splitting theorem to Nijenhuis operator with constant eigenvalues, we get that the question is reduced to the case of single constant eigenvalue.

As $L - \lambda \, \mathrm{Id}$ is Nijenhuis for constant λ, we get that the question is reduced to the case of **nilpotent Nijenhuis operators**.

Nilpotent elements: facts from linear algebra

Linear algebra: given nilpotent operator we have two natural flags of subspaces

$$TM \supset \operatorname{Im} L \supset \operatorname{Im} L^2 \supset \cdots \supset \operatorname{Im} L^{s-1} \supset \operatorname{Im} L^s = \{0\},$$

and

$$\operatorname{Ker} L \subset \operatorname{Ker} L^2 \subset \cdots \subset \operatorname{Ker} L^{s-1} \subset \operatorname{Ker} L^s = TM.$$

Here $L^{s-1} \neq 0$. The quantity s is called **the height of nilpotent operator** L. The Jordan structure of L is uniquely reconstructed from these flags.

For example, for operator of height s all the blocks are dimension of $s-1$ and lower. The number of $s-1$-dimensional blocks is the dimension of $\operatorname{Im} L^{s-1}$ or codimension of $\operatorname{Ker} L^{s-1}$.

From previous lectures we know that for Nijenhuis operator $\operatorname{Im} L^k$ are all integrable distributions. For kernels this is not the case.

Kobayashi's example

Consider $\mathbb{R}^4$ with coordinates x^i, $i = 1 \dots 4$. Take the following vector fields:

$$\xi_1 = \frac{\partial}{\partial x^1}, \quad \xi_2 = \frac{\partial}{\partial x^2}, \quad \xi_3 = \frac{\partial}{\partial x^3}, \quad \xi_4 = \frac{\partial}{\partial x^4} + (1 + x^3)\frac{\partial}{\partial x^1}.$$

Define operator L acting as follows $L\xi_1 = \xi_2$, $L\xi_i = 0$, $i = 2 \dots 4$. The matrix of this operator is

$$L = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 - x^3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The vectors ξ_i's span the tangent space. The $\operatorname{Im} L = \operatorname{span}\langle \partial_{x^2} \rangle$. Thus, $[L\xi_i, \xi_j] = 0$, $[L\xi_i, L\xi_j] = 0$ and $L^2 = 0$. We get that the Nijenhuis torsion vanishes. At the same time the kernel is spanned by ξ_2, ξ_3, ξ_4 and $[\xi_3, \xi_4] = \xi_1$ and is not integrable.

Thompson theorem

For constant operator all the images and all the kernels are integrable. This is the necessary condition. It turns out it is sufficient.

Theorem (Thompson)

Nijenhuis nilpotent operator with locally constant Jordan structure can be brought to constant form if and only if all the distrubutions $\operatorname{Ker} L^k$ *are integrable.*

Remark: The original work is written in the language of $G-$structures. Basically, that is a subbundle of the frame tangent bundle with structure group $G \subset GL(n)$. In our case the natural structure group is the one, preserving the Jordan normal form of an operator at each point. If this Jordan normal form does not change from point to point, then such $G-$structure is called $0-$deformable. In original work the Thompson theorem deals with the integrability of such structures.

Proof of Thompson theorem: step 1

Consider $s = 2$ or, in other words, L is non-zero and $L^2 = 0$. Assume, that rank of L is m and k is the dimension of kernel. Note that normal Jordan form of L consists of Jordan blocks of dimension two and one. We have that k equals to the number of both Jordan blocks of sizes two and one, while m is the number of blocks of dimension two. Thus $k \geq m$.

As kernel is integrable, we get that we can choose coordinates

$$x^1, \dots, x^k, y^1, \dots, y^m,$$

where ∂_{x^i} span the kernel. We get that operator takes form

$$L = \begin{pmatrix} 0_k & M \\ 0 & 0_m \end{pmatrix}$$

Here matrix M is $k \times m$ matrix of rank m. The vanishin of the Nijenhuis torsion yields

$$0 = -L[M_i^\alpha \partial_{x^\alpha}, \partial_{y^j}] - L[\partial_{y^i}, M_j^\alpha \partial_{x^\alpha}] + [M_i^\alpha \partial_{x^\alpha}, M_j^\alpha \partial_{x^\alpha}] = [M_i^\alpha \partial_{x^\alpha}, M_j^\alpha \partial_{x^\alpha}].$$

Proof of Thompson theorem: step 2

Denote $\xi_i = L\partial_{y^i}$. We have that $[\xi_i, \xi_j] = 0$ and $\mathcal{L}_{\xi_i} y^j = 0$. There exist functions $f^1, \dots, f^m$, such that $\mathcal{L}_{\xi^i} f^j = \delta_i^j$ and, by definition

$$L^* \mathrm{d}\, f^i = \mathrm{d}\, y^j.$$

We take new coordinates $\bar{x}^i = f^i, \bar{y}^i = y^i, i = 1, \dots, m$ and $\bar{z}^j = g^j, j = 1, \dots, n-2m$. Here g_j are arbitrary integrals of the distribution $\operatorname{Ker} L$ that are functionally independent with the rest of the integrals. In given coordinates

$$L = \begin{pmatrix} 0_m & 0 & \mathrm{Id}_m \\ 0 & 0_{k-m} & 0 \\ 0 & 0 & 0_m \end{pmatrix},$$

where 0_p is zero matrix of dimension $p \times p$ and Id_m is identity matrix of size $m \times m$.

Proof of Thompson theorem: step 3

Assume that L is of height $s+1$. First, take the coordinates $x^1, \dots, x^k, y^1, \dots, y^m$, adapted to the distribution $\operatorname{Ker} L$. We get that L is in the form

$$L = \begin{pmatrix} 0_k & M \\ 0 & L_2 \end{pmatrix}.$$

Let us calculate the Nijenhuis torsion. We get

$$\begin{aligned}\mathcal{N}_L(\partial_{x^i}, \partial_{y^j}) &= -L[L\partial_{x^i}, \partial_{y^j}] - L[\partial_{x^i}, L\partial_{y^j}] + [L\partial_{x^i}, L\partial_{y^j}] = \\ &= -L[\partial_{x^i}, L\partial_{y^j}] = -L[\partial_{x^i}, (L_2)^\alpha_j \partial_{y^\alpha} + M^\beta_j \partial_{x^\beta}] = \\ &= -L\left(\frac{\partial (L_2)^\alpha_j}{\partial x^i}\partial_{y^\alpha}\right) = 0.\end{aligned}$$

By definition the rank of the submatrix, consisting of M and L_2 is $n-k$, thus, the last equation implies that $\frac{\partial (L_2)^\alpha_j}{\partial x^i} = 0$ or, in other words, L_2 does not depend on x^i.

Proof of Thompson theorem: step 4

Consider quotient space $M/\mathcal{F}$, where $\mathcal{F}$ is a foliation, corresponding to distribution $\operatorname{Ker} L$. The quotient operator L_2 is Nijenhuis and its height is s. Applying the induction assumption we get that L_2 can be brought to the constant form by only coordinates change in y. Without loss of generality we assume that L_2 is in Jordan normal form

$$L_2 = \begin{pmatrix} J_{r_1} & 0 & \dots & 0 \\ 0 & J_{r_2} & \dots & 0 \\ & & \ddots & \\ 0 & 0 & \dots & J_{r_\nu} \end{pmatrix}$$

Where J_{r_i} is a Jordan block $r_i \times r_i$. We split y coordinates into ν groups with r_i coordinates in $i-$th group

$$y^{1,1}, \dots, y^{1,r_1}, y^{1,2}, \dots, y^{2,r_2}, \dots, y^{1,\nu}, \dots, y^{\nu,r_\nu}.$$

Here $r_1 + \dots + r_\nu = m$.

Proof of Thompson theorem: step 5

We have that in terms of differentials the following holds

$$L^* \mathrm{d}\, y^{j,i} = \mathrm{d}\, y^{j,i+1}, \quad 1 \leq i < r_j,$$
$$L^* \mathrm{d}\, y^{j,r_j} = 0.$$

In terms of vector fields we have

$$L\partial_{y^{j,i}} = \partial_{y^{j,i-1}} + M^{\alpha}_{j,i}\partial_{x^\alpha}, \quad 1 < i \leq r_j,$$
$$L\partial_{y^{j,1}} = M^{\alpha}_{j,1}\partial_{x^\alpha}.$$

The latter implies that for all appropriate indices a, b, i, j:

$$[L\partial_{y^{a,i}}, \partial_{y^{b,j}}] \in \mathrm{span}\langle \partial_{x^1}, \ldots, \partial_{x^k} \rangle = \mathrm{Ker}\, L.$$

Thus, the Nijenhuis torsion takes form

$$\mathcal{N}_L(\partial_{y^{a,i}}, \partial_{y^{b,j}}) = [L\partial_{y^{a,i}}, L\partial_{y^{b,j}}] = 0.$$

Proof of Thompson theorem: step 5

Denote the collection of vector fields $\xi_{a,i} = L\partial_{y^{a,i}}$. From previous we get that these vector fields pairwise commute and linearly independent by definition. Now, by Frobenius 2 theorem, we choose the coordinates. This is how we do it.

1. For each $\xi_{a,1}$ we pick function $f^{a,1}$, such that $\xi_{j,i}(f^{a,1}) = \delta_i^1 \delta_a^j$. This yields ν functions with property

$$L^* \mathrm{d}\, f^{a,1} = \mathrm{d}\, y^{a,1}.$$

2. We take $f^{a,l}$ for $1 \leq l \leq r_a - 1$ to be $y^{a,l-1}$. From earlier formulas we get

$$\xi_{j,i}(f^{a,l}) = \xi_{j,i}(y^{a,l-1}) = \langle L\partial_{y^{j,i}}, \mathrm{d}\, y^{a,l-1} \rangle = \langle \partial_{y^{j,i}}, L^* \mathrm{d}\, y^{a,l-1} \rangle =$$
$$= \langle \partial_{y^{j,i}}, \mathrm{d}\, y^{a,l} \rangle = \delta_i^l \delta_j^a.$$

We also have $L^* \mathrm{d}\, f^{a,l} = \mathrm{d}\, f^{a,l+1}$. In each block this yields $r_a - 1$ functions. All together we got m functions, all of them are linearly independent.

Proof of Thompson theorem: step 5

3. We take ν functions $h^i = y^{j,r_j}$, which are integrals of distribution, spanned by $\xi_{a,i}$'s. Indeed,

$$\xi_{a,i}(y^{j,r_j}) = \langle L\partial_{y^{a,i}}, \mathrm{d}\, y^{j,r_j}\rangle = \langle \partial_{y^{a,i}}, L^*\mathrm{d}\, y^{j,r_j}\rangle = 0.$$

4. We take $k-\nu$ integrals of the distribution, spanned by $\xi_{a,i}$'s, functionally independent with the ones we have chosen in previous step, that is $\mathrm{d}\, y^{j,r_j}$. We denote them as $g^1, \dots, g^{k-\nu}$.

Gathering all the functions together, we get the following collection of conditions

$$L^*\mathrm{d}\, f^{a,1} = \mathrm{d}\, y^{a,1}, \quad L^*\mathrm{d}\, f^{a,l} = \mathrm{d}\, f^{a,l+1}, l = 1, \dots, r_a - 1$$
$$L^*\mathrm{d}\, h^i = 0, i = 1, \dots, \nu, \quad L^*\mathrm{d}\, g^j = 0, j = 1, \dots, k-\nu$$

Thus, L^* consists of zeroes and ones and, in particular, is in constant form. The theorem is proved.

Some corollaries

Corollary (Jordan block of maximal size)

Assume that Nijenhuis operator L is similar to the Jordan block of maximal size. Then it can be brought to the constant form.

Proof. For Jordan block of maximal size we have $\operatorname{Ker} L^k = \operatorname{Im} L^{n-k}$. Thus, the kernels are itegrable and we apply Thompson theorem.

Corollary (Direct sum of the Jordan blocks J_m)

Assume that Nijenhuis operator L is similar in each point to the sum of k Jordan blocks of the same dimension m. In particular $n = km$. Then L can be brought to constant form.

Proof. Similar to the previous case we have that $\operatorname{Ker} L^k = \operatorname{Im} L^{m-k}$. Thus, the kernels are integrable and we apply Thompson theorem..

Some corollaries

Recall that $gl-$regular Nijenhuis operator is an operator with the following property: $\mathrm{Id}, L, \dots, L^{n-1}$ are linearly independent. The equivalent definition is the following: for each eigenvalue there is only one Jordan block.

Corollary (Constant form of $gl-$regular Nijenhuis operator)

Consider $gl-$regular Nijenhuis operator L with real eigenvalues. Then the following conditions are equivalent:

- *L can be brought to constant form*
- *The eigenvalues of L are constants (or, equivalently, the coefficients of characteristic polynomial are constants)*

Proof: Applying splitting lemma and Corollary about Jordan block of maximal size does the trick.

Beyond integrability

What happens when the distributions Ker L^k are not integrable?

We study the simplest case, when we have single non-integrable distribution, which means that $L^2 = 0$. This is exactly the algebraic case, we have started the proof of Thompson theorem with.

Normal form of $L^2 = 0$

Theorem (Case $L^2 = 0$)

Let L be a nilpotent Nijenhuis operator such that $L^2 = 0$ and its rank is constant and equals to k. Then there exist coordinates $x^1, \dots, x^k, y^1, \dots, y^m$, such that

$$L = \begin{pmatrix} 0_k & \mathrm{Id_k} & M(y) \\ 0 & 0_k & 0 \\ 0 & 0 & 0_{m-k} \end{pmatrix}, \qquad (1)$$

where 0_i is an $i \times i$ zero matrix, $\mathrm{Id_k}$ is $k \times k$ zero matrix. Here M is $k \times m - k$ matrix that depends only on $y^1, \dots, y^m$. Moreover, ervery matrix in the form (1) is Nijenhuis.

Proof for case $L^2 = 0$

The distribution $\operatorname{Im} L$ is integrable and has dimension k, so we choose coordinates $x^1, \dots, x^k, y^1, \dots, y^m$, adapted to the distribution. L is in the form

$$L = \begin{pmatrix} 0_k & N(x,y) \\ 0 & 0_m \end{pmatrix}$$

The difference with the Thompson theorem is exactly in the size of N: it is $k \times m$ matrix and its rank is k.

Calculating the Nijenhuis torsion we get that vector fields $\xi_i = L\partial_{y^i}$ pairwise commute. Without loss of generailty assume that $\xi_1, \dots, \xi_k$ are linearly independent (otherwise we interchange variables y^i to make this condition to hold).

Taking functions $f^1, \dots, f^k$ for these vector fields as new x^i's and $y^1, \dots, y^m$ as integrals of the distribution, we get new coordinate system.

Proof for case $L^2 = 0$

In new coordinates (which we denote as old ones for simplicity) the matrix of L has the form

$$L = \begin{pmatrix} 0_k & \mathrm{Id_k} & M(x,y) \\ 0 & 0_k & 0 \\ 0 & 0 & 0_{m-k} \end{pmatrix},$$

At the same time $\xi_1 = \partial_{x^1}, \dots, \xi_k = \partial_{x^k}$. For $i = 1, \dots, k, j = k+1, \dots, m$ we get

$$[\xi_i, \xi_j] = [\partial_{x^i}, M_j^\alpha \partial_{x^\alpha}] = \frac{\partial M_j^\alpha}{\partial x^i} \partial_\alpha = 0.$$

Thus, M does not depend on x and Theorem is proved.

Almost normal form

Consider L in the form:

$$L = \begin{pmatrix} 0_k & N(y) \\ 0 & 0_m \end{pmatrix} \tag{2}$$

This is Nijenhuis operator. For arbitrary function g we have

$$L^* \mathrm{d}\, g = N(y)^{\alpha}_{1} \frac{\partial g}{\partial x^{\alpha}} \mathrm{d}\, y^1 + \dots + N(y)^{\alpha}_{m} \frac{\partial g}{\partial x^{\alpha}} \mathrm{d}\, y^m$$

As rank N is k, we have $L^* \mathrm{d}\, g = 0$ iff $\frac{\partial g}{\partial x^i} = 0$ or, in other words g depends only on y. Thus, in coordinate change, preserving (2) we may assume that $\bar{y}^i = y^i$, as this part of coordinate change can be done independently. Thus, we get that $\frac{\partial f}{\partial x^i}$ does not depend on x^i. Thus, the preserving (2) coordinate change is in the form

$$\bar{x}^i = h^i_{\alpha}(y) x^{\alpha} + h^i_0, \quad \bar{y}^j = g^j(y^1, \dots, y^m)$$

Almost normal form

Under these coordinate transformation we get that N is transformed as

$$\bar{N} = H^{-1}(y)N(y)\Big(\frac{\partial g}{\partial y}\Big),$$

where H is matrix that consists of h^i_α. Thus, we see that the rows of H are transformed as a collection of differential forms, defined up to the addition and multiplication by some functions. Thus, they define differential ideal.

The zeroes of these forms define a distribution of dimension $m-k$ on quotient $y^1,\dots,y^m$. We denote it as $\mathcal{D}_N$. The following proposition holds.

Proposition

Two normal forms

$$\begin{pmatrix} 0_k & N(y) \\ 0 & 0_m \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 0_k & \bar{N}(\bar{y}) \\ 0 & 0_m \end{pmatrix}$$

are related by coordinate transformation if and only if the distributions $\mathcal{D}_N$ and $\mathcal{D}_{\bar{N}}$ are related by coordinate transformation.

Results for small dimension

Note: For any $n-k$-dimensional distribution on $n-$dimensional affine space in given coordinates $y^1, \dots, y^m$ we can construct Nijenhuis operator L, such that $L^2 = 0$ and L is defined on the $m+k$-dimensional affine space. To do that we take k independent differential forms, that vanish on the distribution and write (2).

- For $n = 2$ every nilpotent operator can be brought to constant form
- For $n = 3$ we have that if $L^2 \neq 0$, then it is Jordan block of maximal size. Otherwise we get that $L^2 = 0$, $k = 1, m = 2$ and $\mathcal{D}_N$ is one-dimensional. It is integrable, thus, we have that in dimension three any non-zero nilpotent operator can be brought to one of the two forms

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Exercises

1. Consider nilpotent operator L of height s on the space of dimension n and $m_1, \ldots, m_{s-1}$ are the dimensions of $\operatorname{Ker} L, \ldots, \operatorname{Ker} L^{s-1}$. How many of Jordan blocks of each dimension L has in Jordan normal form?
2. Consider nilpotent operator L of height s on the space of dimension n and $m_1, \ldots, m_{s-1}$ are the dimensions of $\operatorname{Im} L, \ldots, \operatorname{Im} L^{s-1}$. How many of Jordan blocks of each dimension L has in Jordan normal form?
3. Assume, that L is similar to the matrix which is the sum of k two-dimensional Jordan blocks and single one-dimensional Jordan block. Prove, that L can be brought to constant form

$$L = \begin{pmatrix} 0_k & \mathrm{Id_k} & 0 \\ 0 & 0_k & 0 \\ 0 & 0 & 0 \end{pmatrix},$$