Nijenhuis Geometry

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Lecture 7: Single Jordan block with non-constant eigenvalue and Complex normal forms

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The case of non-constant eigenvalue

Assume that L has single eigenvalue λ and it is not constant. The operator $L_\lambda = L - \lambda \,\mathrm{Id}$ for Nijenhuis operator L is nilpotent but not Nijenhuis. But one may expect good things from this operator due to the formula we have proved earlier

$$L_\lambda^* \mathrm{d}\,\lambda = (L^* - \lambda \,\mathrm{Id})\mathrm{d}\,\lambda = 0.$$

We have a flag of distributions

$$\mathrm{Im}\, L_\lambda \subseteq \mathrm{Im}\, L_\lambda^2 \subset \dots \subset \mathrm{Im}\, L_\lambda^{s-1} \subset \mathrm{Im}\, L_\lambda^s = \{0\}$$

Here s is the height of corresponding Nijenhuis operator. We will need the following theorem.

Theorem

Assume that Nijenhuis operator L has single eigenvalue and the number and sizes of Jordan blocks do not change from point to point. Then the distributions $\mathrm{Im}\, L_\lambda^p, p = 1, \dots$ for eigenvalue λ are all integrable.

Proof of theorem: step 1

For arbitrary basis vector fields ∂_{x^i} and ∂_{x^j}:

$$[L_\lambda \partial_{x^i}, L_\lambda \partial_{x^j}] = [L\partial_{x^i}, L\partial_{x^j}] - [L\partial_{x^i}, \lambda\partial_{x^j}] - [\lambda\partial_{x^i}, L\partial_{x^j}] + [\lambda\partial_{x^i}, \lambda\partial_{x^j}] =$$
$$= L[L\partial_{x^i}, \partial_{x^j}] + L[\partial_{x^i}, L\partial_{x^j}] - L^2[\partial_{x^i}, \partial_{x^j}] - -(L^* \mathrm{d}\,\lambda)_i \partial_{x^j} - \lambda[L\partial_{x^i}, \partial_{x^j}] +$$
$$+ (L^* \mathrm{d}\,\lambda)_j \partial_{x^i} - \lambda[\partial_{x^i}, L\partial_{x^j}] + \lambda \mathrm{d}\,\lambda_i \partial_{x^j} - \lambda \mathrm{d}\,\lambda_j \partial_{x^i} =$$
$$= L_\lambda[L\partial_{x^i}, \partial_{x^j}] + L_\lambda[\partial_{x^i}, L\partial_{x^j}] + (L^* \mathrm{d}\,\lambda - \lambda \mathrm{d}\,\lambda)_j \partial_{x^i} - (L^* \mathrm{d}\,\lambda - \lambda \mathrm{d}\,\lambda)_i \partial_{x^j}$$

Thus, we get the formula

$$[L_\lambda \partial_{x^i}, L_\lambda \partial_{x^j}] = L_\lambda[L\partial_{x^i}, \partial_{x^j}] + L_\lambda[\partial_{x^i}, L\partial_{x^j}].$$

The vector fields $L_\lambda \partial_{x^i}$ span the distribution $\operatorname{Im} L_\lambda$, thus, by Frobenius theorem the image is integrable.

Proof of theorem: step 2

The distribution $\operatorname{Im} L_\lambda$ is integrable and L-invariant. Denote the corresponding integrable foliation as $\mathcal{F}$. By definition $\operatorname{Im} L_\lambda^2 \subseteq \operatorname{Im} L_\lambda$. Thus, the integrability of the distribution in general is equivalent to the integrability of this distribution on each leaf.

We have that the restriction L_1 of Nijenhuis operator L on arbitrary leaf of $\mathcal{F}$ is Nijenhuis operator. The distribution $\operatorname{Im} L_\lambda^2$ on this leaf becomes $\operatorname{Im}(L_1 - \lambda \operatorname{Id})$. Applying previous result, we get the integrability of the second distribution. Proceeding in a similar way, we see that all the distributions are integrable.

Note, that the theorem holds for arbitrary eigenvalue and arbitrary Nijenhuis operator L, not necessary single eigenvalue case.

Single eigenvalue case

Now assume that L has a single eigenvalue. Taking coordinate system, adapted to the flag of images, we get that operator L is in the form:

$$L = \begin{pmatrix} \lambda \operatorname{Id}_{r_1} & * & \dots & * \\ 0 & \lambda \operatorname{Id}_{r_2} & \dots & * \\ & & \ddots & \\ 0 & 0 & \dots & \lambda \operatorname{Id}_{r_\nu} \end{pmatrix}.$$

Here $r_1 + \cdots + r_\nu = n$, ν is the height of the operator and

$$r_i = \dim \operatorname{Im} L^{\nu - i} - \dim \operatorname{Im} L^{\nu - i + 1}.$$

It turns out that in case of single Jordan block we can get a better normal form.

Theorem for non-constant eigenvalue

Theorem (Local structure in case of single eigenvalue)

Assume that L is similar, at each point, to Jordan block with single eigenvalue. Then in appropriate coordinates $x^1, \dots, x^n$, the operator takes the form:

$$\begin{pmatrix} \lambda(x^n) & 1 & 0 & \dots & 0 & g^1 \\ 0 & \lambda(x^n) & 1 & \dots & 0 & g^2 \\ & & & \ddots & & \\ 0 & 0 & 0 & \dots & \lambda(x^n) & g^{n-1} \\ 0 & 0 & 0 & \dots & 0 & \lambda(x^n) \end{pmatrix},$$

where $g^{n-1} = 1$ and $g^i = -\lambda'(n-i-1)x^i$.

Proof of the theorem: step 1

We start with $n = 2$. In this case, the height of L_λ is 2 and in coordinate system, adopted to the distribution $\operatorname{Im} L_\lambda$, it takes the form:

$$L = \begin{pmatrix} \lambda & g^1 \\ 0 & \lambda \end{pmatrix}$$

The Nijenhuis condition (see Lecture 4) is equivalent to the

$$0 = \lambda \mathrm{d}\,\lambda - \mathrm{d}\,\lambda \begin{pmatrix} \lambda & -g^1 \\ 0 & \lambda \end{pmatrix} = \Big(-g^1 \lambda'_{x^1}, 0 \Big).$$

As L is similar to Jordan block at each point, then $g^1 \neq 0$ and λ depends only on x^2. We are searching for coordinate change in the form $\bar{x}^1 = f(x^1, x^2)$, $\bar{x}^2 = x^2$ so that

$$\begin{pmatrix} f'_{x^1} & f'_{x^2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} \frac{1}{f'_{x^1}} & -\frac{f'_{x^2}}{f'_{x^1}} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \lambda & f'_{x^1} \\ 0 & \lambda \end{pmatrix}$$

Thus, we take $f(x^1, x^2)$ to be the antiderivative of $g^1(x^1, x^2)$ in x^1. It defines coordinate change as $g^1 \neq 0$.

Proof of the theorem: step 2

Assume that the statement holds for n, we need to prove it for $n+1$. Take distribution $\operatorname{Ker} L_\lambda = \operatorname{Im} L_\lambda^{n-1}$ and adopted coordinate system $x, y^1, \dots, y^n$. We have

$$L = \begin{pmatrix} \lambda & M \\ 0 & N \end{pmatrix},$$

where M is $1 \times n$ matrix and N is $n \times n$ matrix. Recall, the formula have obtained earlier:

$$0 = [L_\lambda \partial_x, L_\lambda \partial_{y^j}] = L_\lambda [L\partial_x, \partial_{y^j}] + L_\lambda [\partial_x, L\partial_{y^j}] =$$
$$= L_\lambda [\lambda \partial_x, \partial_{y^i}] + L_\lambda [\partial_x, N^\alpha_j \partial_{y^\alpha} + M_j \partial_x] = L_\lambda \left(\frac{\partial N^\alpha_j}{\partial x} \partial_{y^\alpha} \right)$$

We have that rank of L_λ is n and, thus, the last equation implies, that N does not depend on x. Thus, N defines a quotient operator and it is Nijenhuis on the quotient space.

Proof of the theorem: step 3

Applying the induction assumption, we bring N to its normal form. We get

$$L = \begin{pmatrix} \lambda(y^n) & M_2^1 & M_3^1 & \dots & M_{n-1}^1 & g^1 \\ 0 & \lambda(y^n) & 1 & \dots & 0 & g^2 \\ 0 & 0 & \lambda(y^n) & \dots & 0 & g^3 \\ & & & \ddots & & \\ 0 & 0 & 0 & \dots & 1 & g^{n-1} \\ 0 & 0 & 0 & \dots & \lambda(y^n) & 1 \\ 0 & 0 & 0 & \dots & 0 & \lambda(y^n) \end{pmatrix}$$

That is functions stand on the diagonal, first row and last column. As $\operatorname{Im} L_\lambda$ is an L-invariant distribution, we consider the restriction of the operator onto its leaf. Note that this leaf is $y^n = const$.

Proof of the theorem: step 4

The coordinates on $y^n = const$ are $x, y^1, \ldots, y^{n-1}$. The restriction of L we denote as L_1. It is a Nijenhuis operator and, as $y^n = const$, then the eigenvalues of L_1 are constants. This yields

$$L_{1,\lambda} = \begin{pmatrix} 0 & M \\ 0 & J_{n-2}(\lambda) \end{pmatrix},$$

where $L_{1,\lambda} = L_1 - \lambda \,\mathrm{Id}$, M is $1 \times n-2$ matrix and J_{n-2} is a Jordan block of size $n-2 \times n-2$. The operator $L_{1,\lambda}$ is Nijenhuis on the leaf, so:

$$0 = [L_{1,\lambda}\partial_y^i, L_{1,\lambda}\partial_y^j] - L_{1,\lambda}[L_{1,\lambda}\partial_{y^i}] - L_{1,\lambda}[\partial_{y^i}, L_{1,\lambda}\partial_{y^j}] = [L_{1,\lambda}\partial_y^i, L_{1,\lambda}\partial_y^j]$$

Thus, the vector fields $\xi_i = L_{1,\lambda}\partial_{y^i}$ pairwise commute. Following the procedure, described in Lecture 6, we get that the functions y^{n-1} is an integral of the distribution and there exists a function f, such that $f^1 = f, f^2 = y^1, \ldots, f^{n-1} = y^{n-2}$ define the functions $\xi_i(f^j) = \delta_i^j$.

Proof of the theorem: step 5

We get that

$$L_{1,\lambda}^* \mathrm{d} f = \mathrm{d} y^1,$$
$$L_{1,\lambda}^* \mathrm{d} y^i = \mathrm{d} y^{i+1}, i = 1, \dots, n-2,$$
$$L_{1,\lambda}^* \mathrm{d} y^{n-1} = 0.$$

The function $f(x, y^1, \dots, y^n)$ depends on y^n as a parameter of the leaf and is defined up to addition of a function, depending on single variable. We take arbitrary function of this type and going back to the original manifold, we get

$$L_\lambda^* \mathrm{d} f = \mathrm{d} y^1 + \left(g^1 \frac{\partial f}{\partial x} + g^2 \frac{\partial f}{\partial y^1} + \dots + g^n \frac{\partial f}{\partial y^{n-1}} \right) \mathrm{d} y^n,$$
$$L_\lambda^* \mathrm{d} y^i = \mathrm{d} y^{i+1} + g^{i+1} \mathrm{d} y^n, i = 1, \dots, n-1,$$
$$L_\lambda^* \mathrm{d} y^n = 0.$$

Here we used that $g^n = 1$ in normal form.

Proof of the theorem: step 5

Functions $f, y^1, \dots, y^n$ are linearly independent and define new coordinate system. Moreover, all g_i in new coordinates stay canonical, except, may be, the element in the upper right corner.

In new coordinates $y^0, y^1, \dots, y^n$ we have

$$L = \begin{pmatrix} \lambda(y^n) & 1 & 0 & \dots & 0 & g^0 \\ 0 & \lambda(y^n) & 1 & \dots & 0 & g^1 \\ 0 & 0 & \lambda(y^n) & \dots & 0 & g^2 \\ & & & \ddots & & \\ 0 & 0 & 0 & \dots & 1 & g^{n-2} \\ 0 & 0 & 0 & \dots & \lambda(y^n) & g^{n-1} \\ 0 & 0 & 0 & \dots & 0 & \lambda(y^n) \end{pmatrix}$$

Here g^0 is the only component that might be not in the canonical form.

Proof of the theorem: step 6

Vanishing of $\mathcal{N}_L(\partial_{y^0}, \partial_{y^n})$ yields

$$
\begin{aligned}
0 &= [L\partial_{y^0}, L\partial_{y^n}] - L[L\partial_{y^0}, \partial_{y^n}] - L[\partial_{y^0}, L\partial_{y^n}] = \\
&= [\lambda\partial_{y^0}, g^\alpha\partial_{y^\alpha}] - L[\lambda\partial_{y^0}, \partial_{y^n}] - L[\partial_{y^0}, g^\alpha\partial_{y^\alpha}] = \\
&= \lambda\frac{\partial g^0}{\partial y^0}\partial_{y^0} - \lambda\lambda'\partial_{y^0} + \lambda\lambda'\partial_{y^0} + \lambda\lambda'\partial_{y^0} - \lambda\frac{\partial g^0}{\partial y^0}\partial^0 = 0.
\end{aligned}
$$

Vanishing of $\mathcal{N}_L(\partial_{y^1}, \partial_{y^n})$ yields

$$
\begin{aligned}
0 &= [L\partial_{y^1}, L\partial_{y^n}] - L[L\partial_{y^1}, \partial_{y^n}] - L[\partial_{y^1}, L\partial_{y^n}] = \\
&= [\lambda\partial_{y^1} + \partial_{y^0}, g^\alpha\partial_{y^\alpha}] - L[\lambda\partial_{y^1} + \partial_{y^0}, \partial_{y^n}] - L[\partial_{y^1}, g^\alpha\partial_{y^\alpha}] = \\
&= \frac{\partial g^0}{\partial y^0}\partial_{y^0} + \lambda\frac{\partial g^0}{\partial y^1}\partial_{y^0} + \lambda\frac{\partial g^1}{\partial y^1}\partial_{y^1} - \lambda\lambda'\partial_{y^1} + \lambda'\lambda\partial_{y^1} + \lambda'\partial_{y^0} - \\
&- \lambda\frac{\partial g^0}{\partial y^1}\partial_{y^0} - \lambda\frac{\partial g^1}{\partial y^1}\partial_{y^1} - \frac{\partial g^1}{\partial y^1}\partial_{y^0} = \left(\frac{\partial g^0}{\partial y^0} + \lambda' - \frac{\partial g^1}{\partial y^1}\right)\partial_{y^0}
\end{aligned}
$$

Proof of the theorem: step 6

The equation implies that $\frac{\partial g^0}{\partial y^0} = -(n-1)\lambda'$. Now, vanishing of $\mathcal{N}_L(\partial_{y^i}, \partial_{y^n})$ for $i = 2, \dots, n-1$ yields

$$
\begin{aligned}
0 &= [L\partial_{y^i}, L\partial_{y^n}] - L[L\partial_{y^i}, \partial_{y^n}] - L[\partial_{y^i}, L\partial_{y^n}] = \\
&= [\lambda\partial_{y^i} + \partial_{y^{i-1}}, g^\alpha \partial_{y^\alpha}] - L[\lambda\partial_{y^i} + \partial_{y^{i-1}}, \partial_{y^n}] - L[\partial_{y^i}, g^\alpha \partial_{y^\alpha}] = \\
&= \frac{\partial g^0}{\partial y^{i-1}}\partial_{y^0} + \frac{\partial g^{i-1}}{\partial y^{i-1}}\partial_{y^{i-1}} + \lambda\frac{\partial g^0}{\partial y^i}\partial_{y^0} + \lambda\frac{\partial g^i}{\partial y^i}\partial_{y^i} - \lambda\lambda'\partial_{y^i} + \\
&+ \lambda\lambda'\partial_{y^i} + \lambda'\partial_{y^{i-1}} - \lambda\frac{\partial g^0}{\partial y^i}\partial_{y^0} - \lambda\frac{\partial g^i}{\partial y^i}\partial_{y^i} - \frac{\partial g^i}{\partial y^i}\partial_{y^{i-1}} = \\
&= \left(\frac{\partial g^{i-1}}{\partial y^{i-1}} + \lambda' - \frac{\partial g^i}{\partial y^i}\right)\partial_{y^{i-1}} + \frac{\partial g^0}{\partial y^{i-1}}\partial_{y^0}.
\end{aligned}
$$

This implies that $\frac{\partial g^0}{\partial y^{i-1}} = 0$ for $i = 2, \dots, n-1$. Together with previous calculations we get $g^0 = -(n-1)\lambda' y^0 + h(y^{n-1}, y^n)$.

Proof of the theorem: step 7

The final step is the search for a coordinate change in the form $x^0 = y^0 + r(y^{n-1}, y^n), x^i = y^i$ for $i = 1, \ldots, n$, such that

$$L^* \mathrm{d}\, x^0 = \lambda \mathrm{d}\, x^0 + \mathrm{d}\, x^1 - (n-1)\lambda' x^0 \mathrm{d}\, x^n$$

Using L^*_λ we get $L^*_\lambda \mathrm{d}\, x^0 = \mathrm{d}\, x^1 - (n-1)\lambda' x^0 \mathrm{d}\, x^n$. This yields

$$\mathrm{d}\, x^1 + \Big(\frac{\partial r}{\partial x^{n-1}} - (n-1)\lambda'(x^0 - r) + h\Big)\mathrm{d}\, x^n = \mathrm{d}\, x^1 - (n-1)\lambda' x^0 \mathrm{d}\, x^n.$$

Here we used, that $x^{n-1} = y^{n-1}, x^n = y^n$. Finally, we get the equation

$$\frac{\partial r}{\partial x^{n-1}} + (n-2)\lambda' r + h = 0$$

This is parametric ODE, disguised as PDE. It has the solution for arbitrary h. Thus, the theorem is proved.

Frolicher-Nijenhuis bracket

Consider operator fields L and R. The Frolicher-Njienhuis bracket of these operator fields is defined as

$$[[L, R]](\xi, \eta) = \mathcal{N}_{L+R}(\xi, \eta) - \mathcal{N}_L(\xi, \eta) - \mathcal{N}_R(\xi, \eta).$$

In terms of commutators the formula is

$$\begin{aligned}[[L, R]](\xi, \eta) = L[R\xi, \eta] + L[\xi, R\eta] + R[L\xi, \eta] + R[\xi, L\eta] - \\ - LR[\xi, \eta] - RL[\xi, \eta] - [R\xi, L\eta] - [L\xi, R\eta]\end{aligned}$$

The obvious properties are

- $[[L, R]]$ is a tensor of type $(1, 2)$, skew-symmetric in lower indices
- $[[L, L]] = 2\mathcal{N}_L$
- $[[L, R]] = [[R, L]]$

Frolicher-Nijenhuis bracket

The next formula relates the multiplication of matrices and Frolicher-Nijenhuis bracket. The following Proposition holds.

Proposition

For arbitrary operator fields M, L, R and arbitrary vector fields ξ, η the formula holds

$$[[LM, R]](\xi, \eta) + [[L, RM]](\xi, \eta) - [[L, R]](M\xi, \eta) - [[L, R]](\xi, M\eta) = \\ = L[[M, R]](\xi, \eta) + R[[M, L]](\xi, \eta)$$

Proof of proposition

$$
\begin{aligned}
[[LM,R]](\xi,\eta) &= LM[R\xi,\eta] + LM[\xi,R\eta] + R[LM\xi,\eta] + R[\xi,LM\eta] - \\
&\quad - LMR[\xi,\eta] - RLM[\xi,\eta] - [R\xi,LM\eta] - [LM\xi,R\eta] \\
[[L,RM]](\xi,\eta) &= L[RM\xi,\eta] + L[\xi,RM\eta] + RM[L\xi,\eta] + RM[\xi,L\eta] - \\
&\quad - LRM[\xi,\eta] - RML[\xi,\eta] - [RM\xi,L\eta] - [L\xi,RM\eta] \\
-[[L,R]](M\xi,\eta) &= -L[RM\xi,\eta] - L[M\xi,R\eta] - R[LM\xi,\eta] - R[M\xi,L\eta] + \\
&\quad + LR[M\xi,\eta] + RL[M\xi,\eta] + [RM\xi,L\eta] + [LM\xi,R\eta] \\
-[[L,R]](\xi,M\eta) &= -L[R\xi,M\eta] - L[\xi,RM\eta] - R[L\xi,M\eta] - R[\xi,LM\eta] + \\
&\quad + LR[\xi,M\eta] + RL[\xi,M\eta] + [R\xi,LM\eta] + [L\xi,RM\eta], \\
-L[[M,R]](\xi,\eta) &= -LM[R\xi,\eta] - LM[\xi,R\eta] - LR[M\xi,\eta] - LR[\xi,M\eta] + \\
&\quad + LMR[\xi,\eta] + LRM[\xi,\eta] + L[R\xi,M\eta] + L[M\xi,R\eta], \\
-R[[M,L]](\xi,\eta) &= -RL[M\xi,\eta] - RL[\xi,M\eta] - RM[L\xi,\eta] - RM[\xi,L\eta] + \\
&\quad + RLM[\xi,\eta] + RML[\xi,\eta] + R[M\xi,L\eta] + R[L\xi,M\eta]
\end{aligned}
$$

Frolicher-Nijenhuis bracket

Proposition

For a Nijenhuis operator L and arbitrary functions f, g, satisfying Margelyan's condition, we have $[[f(L), g(L)]] = 0$.

Proof: It is enough to prove this statement for powers of L, that is for arbitrary k, m we have that $[[L^k, L^m]] = 0$. First, we take $L = R$ and $M = L^k$. As $[[L, L]] = 0$ the formula for Frolicher-Nijenhuis bracket yields

$$[[L^{k+1}, L]](\xi, \eta) = L[[L^k, L]](\xi, \eta).$$

This implies, that $[[L^k, L]](\xi, \eta) = L^{k-1}[[L, L]](\xi, \eta) = 0$.

Now take $M = L^k, R = L^m$. We get

$$[[L^{k+1}, L^m]](\xi, \eta) = L[[L^k, L^m]](\xi, \eta).$$

Similar to the previous, this yields that $[[L^k, L^m]] = 0$

Complex eigenvalues

Consider $\mathbb{C}^n$ which can be treated as $\mathbb{R}^{2n}$. Given basis $\eta_1, \ldots, \eta_n \in \mathbb{C}^n$, we get a real basis

$$\eta_1, \ldots, \eta_n, \xi_1 = \mathrm{i}\eta_1, \ldots, \xi_n = \mathrm{i}\eta_n.$$

In this cases the operator J of multiplication by i takes the form

$$\mathrm{J} = \begin{pmatrix} 0 & -\mathrm{Id}_n \\ \mathrm{Id}_n & 0 \end{pmatrix},$$

where Id_n is the $n \times n$ identity matrix. The real operators $L : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$, which "were" complex, are exactly the operators $L\mathrm{J} - \mathrm{J}L = 0$. They have the form

$$L = \begin{pmatrix} A & -B \\ B & A \end{pmatrix}.$$

In this case, the operator can be written as n-dimensional operator $L^{\mathbb{C}} = A + \mathrm{i}B$. That is $(L^{\mathbb{C}})^i_j = A^i_j + \mathrm{i}B^i_j$.

Complex eigenvalues

Theorem

Let L be a Nijenhuis operator on M with no real eigenvalues, i.e., its spectrum at every point $x \in M$ belongs to $\mathbb{C} \setminus \mathbb{R}$. Then

1. *M possess natural complex structure, compatible with L*
2. *$L^{\mathbb{C}}$ is a complex holomorphic tensor field on M w.r.t. J, that is the components of the operator $(L^{\mathbb{C}})^i_j$ are holomorphic functions of z^i.*
3. *The complex Nijenhuis tensor of L vanishes, i.e.*

$$(L^{\mathbb{C}})^\alpha_j \frac{\partial (L^{\mathbb{C}})^i_k}{\partial z^\alpha} - (L^{\mathbb{C}})^\alpha_k \frac{\partial (L^{\mathbb{C}})^i_j}{\partial z^\alpha} - (L^{\mathbb{C}})^i_\alpha \frac{\partial (L^{\mathbb{C}})^\alpha_k}{\partial z^j} + (L^{\mathbb{C}})^i_\alpha \frac{\partial (L^{\mathbb{C}})^\alpha_j}{\partial z^k} = 0.$$

Proof of Theorem: step 1

In canonical coordinates $x^1, \dots, x^n, y^1, \dots, y^n$ of the complex structure $\mathrm{J} = f(L)$, the Nijenhuis operator L takes the form

$$L = \begin{pmatrix} A & -B \\ B & A \end{pmatrix}.$$

The entries of the complexified version of this operator are $A^\alpha_i(x, y) + \mathrm{i} B^\alpha_i(x, y)$. For these entries to be analytic functions, we need the Cauchy-Riemann conditions to hold. In multi-dimensional case this implies

$$\frac{\partial A^\alpha_i}{\partial x^j} - \frac{\partial B^\alpha_i}{\partial y^j} = 0,$$
$$\frac{\partial A^\alpha_i}{\partial y^j} + \frac{\partial B^\alpha_i}{\partial x^j} = 0.$$

Proof of Theorem: step 2

We have that $[[L,\mathrm{J}]] = [[L, f(L)]] = 0$. For $\partial_{x^i}, \partial_{y^j}$ this yields

$$[[L,\mathrm{J}]](\partial_{x^i}, \partial_{y^j}) = \mathrm{J}[L\partial x^i, \partial_{y^j}] + \mathrm{J}[\partial_{x^i}, L\partial_{y^j}] + [L\partial_{x^i}, \partial_{x^j}] - [\partial_{y^i}, L\partial_{y^j}] =$$

$$= \mathrm{J}[A_i^\alpha \partial_{x^\alpha} + B_i^\beta \partial_{y^\beta}, \partial_{y^j}] + \mathrm{J}[\partial_{x^i}, -B_j^\alpha \partial_{x^\alpha} + A_j^\beta \partial_{y^\beta}] +$$

$$+ [A_i^\alpha \partial_{x^\alpha} + B_i^\beta \partial_{y^\beta}, \partial_{x^j}] - [\partial_{y^i}, -B_j^\alpha \partial_{x^\alpha} + A_j^\beta \partial_{y^\beta}] =$$

$$= \mathrm{J}\Big(- \frac{\partial A_i^\alpha}{\partial y^j} \partial_{x^\alpha} - \frac{\partial B_i^\beta}{\partial y^j} \partial_{y^\beta} \Big) + \mathrm{J}\Big(- \frac{\partial B_j^\alpha}{\partial x^i} \partial_{x^\alpha} + \frac{\partial A_j^\beta}{\partial x^i} \partial_{y^\beta} \Big) -$$

$$- \frac{\partial A_i^\alpha}{\partial x^j} \partial_{x^\alpha} - \frac{\partial B_i^\beta}{\partial x^j} \partial_{y^\beta} + \frac{\partial B_j^\alpha}{\partial y^i} \partial_{x^\alpha} - \frac{\partial A_j^\beta}{\partial y^i} \partial_{y^\beta} =$$

$$= \Big(- \frac{\partial A_i^\beta}{\partial y^j} - \frac{\partial B_j^\beta}{\partial x^i} - \frac{\partial A_j^\beta}{\partial y^i} - \frac{\partial B_i^\beta}{\partial x^j} \Big) \partial_{y^\beta} +$$

$$+ \Big(\frac{\partial B_i^\alpha}{\partial y^j} - \frac{\partial A_j^\alpha}{\partial x^i} - \frac{\partial A_i^\alpha}{\partial x^j} + \frac{\partial B_j^\alpha}{\partial y^i} \Big) \partial_{x^\alpha}$$

Proof of Theorem: step 2

For $\partial_{x^i}, \partial_{x^j}$ (here $i \neq j$) we get

$$
\begin{aligned}
[[L, \mathrm{J}]](\partial_{x^i}, \partial_{x^j}) &= \mathrm{J}[L\partial x^i, \partial_{x^j}] + \mathrm{J}[\partial_{x^i}, L\partial_{x^j}] + [L\partial_{x^i}, \partial_{y^j}] - [\partial_{y^i}, L\partial_{x^j}] = \\
&= \mathrm{J}[A_i^\alpha \partial_{x^\alpha} + B_i^\beta \partial_{y^\beta}, \partial_{x^j}] + \mathrm{J}[\partial_{x^i}, A_j^\alpha \partial_{x^\alpha} + B_j^\beta \partial_{y^\beta}] - \\
&- [A_i^\alpha \partial_{x^\alpha} + B_i^\beta \partial_{y^\beta}, \partial_{y^j}] - [\partial_{y^i}, A_j^\alpha \partial_{x^\alpha} + B_j^\beta \partial_{y^\beta}] = \\
&= \mathrm{J}\Big(- \frac{\partial A_i^\alpha}{\partial x^j} \partial_{x^\alpha} - \frac{\partial B_i^\beta}{\partial x^j} \partial_{y^\beta} \Big) + \mathrm{J}\Big(\frac{\partial A_j^\alpha}{\partial x^i} \partial_{x^\alpha} + \frac{\partial B_j^\beta}{\partial x^i} \partial_{y^\beta} \Big) + \\
&+ \frac{\partial A_i^\alpha}{\partial y^j} \partial_{x^\alpha} + \frac{\partial B_i^\beta}{\partial y^j} \partial_{y^\beta} - \frac{\partial A_j^\alpha}{\partial y^i} \partial_{x^\alpha} - \frac{\partial B_j^\beta}{\partial y^i} \partial_{y^\beta} = \\
&= \Big(\frac{\partial B_i^\alpha}{\partial x^j} - \frac{\partial B_j^\alpha}{\partial x^i} + \frac{\partial A_i^\alpha}{\partial y^j} - \frac{\partial A_j^\alpha}{\partial y^i} \Big) \partial_{x^\alpha} + \\
&+ \Big(- \frac{\partial A_i^\beta}{\partial x^j} + \frac{\partial A_j^\beta}{\partial x^i} + \frac{\partial B_i^\beta}{\partial y^j} - \frac{\partial B_j^\beta}{\partial y^j} \Big) \partial_{y^\beta}
\end{aligned}
$$

Proof of Theorem: step 3

Renaming the upper indices we gather the following system

$$\begin{aligned}
&\left(\frac{\partial A_i^\alpha}{\partial y^j} + \frac{\partial B_i^\alpha}{\partial x^j}\right) + \left(\frac{\partial A_j^\alpha}{\partial y^i} + \frac{\partial B_j^\alpha}{\partial x^i}\right) = 0,\\
&\left(\frac{\partial A_i^\alpha}{\partial x^j} - \frac{\partial B_i^\alpha}{\partial y^j}\right) + \left(\frac{\partial A_j^\alpha}{\partial x^i} - \frac{\partial B_j^\alpha}{\partial y^i}\right) = 0,\\
&\left(\frac{\partial A_i^\alpha}{\partial y^j} + \frac{\partial B_i^\alpha}{\partial x^j}\right) - \left(\frac{\partial A_j^\alpha}{\partial y^i} + \frac{\partial B_j^\alpha}{\partial x^i}\right) = 0, \quad i \neq j\\
&\left(\frac{\partial A_i^\alpha}{\partial x^j} - \frac{\partial B_i^\alpha}{\partial y^j}\right) - \left(\frac{\partial A_j^\alpha}{\partial x^i} - \frac{\partial B_j^\alpha}{\partial y^i}\right) = 0, \quad i \neq j
\end{aligned}$$

For $i = j$ from the first two equations we get Cauchy-Riemann conditions on $A_i^\alpha + \mathrm{i}B_i^\alpha$ with respect to $z^i = x^i + \mathrm{i}y^i$. For $i \neq j$ we get the Cauchy-Riemann conditions with respect to other variables. Thus, each component of the complexified operator is, in fact, holomorphic with respect to all coordinates.

Proof of Theorem: step 4

Complex coordinate vector fields are $\partial_{z^i} = \frac{1}{2}\big(\partial_{x^i} - \mathrm{i}\partial_{y^i}\big)$. By definition

$$L^{\mathbb{C}}\partial_{z^i} = \frac{1}{2}(A + \mathrm{i}B)(\partial_{x^i} - \mathrm{i}\partial_{y^i}) = \frac{1}{2}L\partial_{x^i} - \frac{1}{2}\mathrm{i}L\partial_{y^i}.$$

The Nijenhuis torsion in complex case takes the form

$$\begin{aligned}
&4\Big(L^{\mathbb{C}}[L^{\mathbb{C}}\partial_{z^i}, \partial_{z^j}] + L^{\mathbb{C}}[\partial_{z^i}, L^{\mathbb{C}}\partial_{z^j}] - [L^{\mathbb{C}}\partial_{z^i}, L^{\mathbb{C}}\partial_{z^j}]\Big) = \\
&= L^{\mathbb{C}}\Big([L\partial_{x^i}, \partial_{x^j}] + [L\partial_{y^i}, \partial_{y^j}] - \mathrm{i}[L\partial_{y^i}, \partial_{x^j}] - \mathrm{i}[L\partial_{x^i}, \partial_{y^j}]\Big) + \\
&+ L^{\mathbb{C}}\Big([\partial_{x^i}, L\partial_{x^j}] + [\partial_{y^i}, L\partial_{y^j}] - \mathrm{i}[\partial_{y^i}, L\partial_{x^j}] - \mathrm{i}[\partial_{x^i}, L\partial_{y^j}]\Big) + \\
&+ \Big(-[L\partial_{x^i}, L\partial_{x^j}] - [L\partial_{y^i}, L\partial_{y^j}] + \mathrm{i}[L\partial_{x^i}, L\partial_{y^j}] + \mathrm{i}[L\partial_{y^i}, L\partial_{x^j}]\Big) = \\
&= \mathcal{N}_L(\partial_{x^i}, \partial_{x^j}) + \mathcal{N}_L(\partial_{y^i}, \partial_{y^j}) - \mathrm{i}\mathcal{N}_L(\partial_{x^i}, \partial_{y^j}) - \mathrm{i}\mathcal{N}_L(\partial_{y^i}, \partial_{x^j})
\end{aligned}$$

Thus, the complex Nijenhuis torsion vanishes. Theorem is proved.

Complex eigenvalues

Linear algebra: For a real operator L, the Jordan block corresponding to a pair of complex conjugate eigenvalues $\mu = a + \mathrm{i}b, \bar{\mu} = a - \mathrm{i}b$ is

$$\begin{pmatrix} \Lambda & \mathrm{Id}_2 & 0_2 & \dots & 0_2 \\ 0_2 & \Lambda & \mathrm{Id}_2 & \dots & 0_2 \\ 0_2 & 0_2 & \Lambda & \dots & 0_2 \\ & & & \ddots & \\ 0_2 & 0_2 & 0_2 & \dots & \Lambda \end{pmatrix},$$

where

$$0_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}, \quad \mathrm{Id}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The dimension is even.

Complex normal forms

Applying the Frobenius theorem for complex distributions, we get the complex analogs of the normal forms we have obtained earlier for case of real eigenvalues.

Theorem (Complex Nijenhuis theorem)

If L has no real eigenvalues and all complex eigenvalues are pairwise distinct, then in an appropriate coordinate system $x^1, y^1, \dots, x^n, y^n$ the operator can be brought to the form

$$L = \begin{pmatrix} \Lambda_1 & 0_2 & \dots & 0_2 \\ 0_2 & \Lambda_2 & \dots & 0_2 \\ & & \ddots & \\ 0_2 & 0_2 & \dots & \Lambda_n \end{pmatrix},$$

where

$$\Lambda_i = \begin{pmatrix} a_i(x^i, y^i) & -b_i(x^i, y^i) \\ b_i(x^i, y^i) & a_i(x^i, y^i) \end{pmatrix}$$

and a_i, b_i satisfy the Cauchy-Riemann conditions.

Complex normal forms

Theorem (Complex diagonal normal form)

Consider L to be a smooth Nijenhuis operator, such that

1. *L has no real eigenvalues*
2. *Complex eigenvalues are pairwise distinct and non-constant*

Then there exits a unique normal form, in which $a_i = \mathfrak{Re}(c_i + (z_i)^{k_i}), b_i = \mathfrak{Im}(c_i + (z_i)^{k_i})$ for complex constant c_i and $z_i = x_i + \mathrm{i} y_i$. In particular, the normal form is polynomial.

Note that in the case of non-real eigenvalues the polynomial form of the operator exists even in smooth case (for real eigenvalues we had to assume the analyticity). Another important feature is the absence of sign parameter δ.

Complex normal forms

Theorem

Assume that L is similar, at each point, to the real Jordan block, corresponding to a pair of complex conjugate eigenvalues. Then in an appropriate coordinate system $x^1, y^1, \dots, x^n, y^n$

$$L = \begin{pmatrix} \Lambda & \mathrm{Id}_2 & 0_2 & \dots & 0_2 & G^1 \\ 0_2 & \Lambda & \mathrm{Id}_2 & \dots & 0_2 & G^2 \\ & & & \ddots & & \\ 0_2 & 0_2 & 0_2 & \dots & \Lambda & G^{n-1} \\ 0_2 & 0_2 & 0_2 & \dots & 0 & \Lambda \end{pmatrix},$$

where for a, b *satisfying the Cauchy-Riemann conditions we have*

$$\Lambda = \begin{pmatrix} a(x^n, y^n) & -b(x^n, y^n) \\ b(x^n, y^n) & a(x^n, y^n) \end{pmatrix}, \quad G^{n-1} = \mathrm{Id}_2,$$

$$G^i = -(n-i-1) \begin{pmatrix} \frac{\partial a}{\partial x^n} & -\frac{\partial a}{\partial y^n} \\ \frac{\partial a}{\partial y^n} & \frac{\partial a}{\partial x^n} \end{pmatrix} \begin{pmatrix} x^i & -y^i \\ y^i & x^i \end{pmatrix}$$