Nijenhuis Geometry

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Lecture 8:
Differentially non-degenerate singular points and global theorems on Nijenhuis operators

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Regular and singular points of Nijenhuis operators

Point p is **algebraically generic for Nijenhuis operator** L if the structure and number of Jordan blocks (the so-called Segre characteristic) does not change in some sufficiently small neighbourhood $U(\mathrm{p})$. We see that the local Nijenhuis geometries in algebraically generic points can be wild (case $L^2 = 0$).

We have studied the following "normal" cases:

- The case of constant normal form
- The case of diagonal normal form with non-constant eigenvalues
- Single Jordan block with real eigenvalue
- Single Jordan block with a pair of complex conjugate eigenvalues

Differentially non-degenerate Nijenhuis operators

The next step in the study of Nijenhuis operator is to study singular points - the points, which are not algebraically generic. We have several classes of such points, but we start with the simplest case.

Consider Nijenhuis operator L with characteristic polynomial

$$\chi_L(t) = \det(t\,\mathrm{Id} - L)^n = t^n - \sigma_1 t^{n-1} - \dots - \sigma_n.$$

We say that the point p is differentially non-degenerate if $\mathrm{d}\,\sigma_i$ are all independent at this point (and, thus, in the entire neighbourhood).

Example: The following operator is Nijenhuis and differentially non-degenerate on the entire plane:

$$L = \begin{pmatrix} x & 1 \\ y & 0 \end{pmatrix}$$

Differentially non-degenerate Nijenhuis operators

The characteristic polynomial is

$$\chi_L(t) = t^2 - xt - y$$

The discriminant of the characteristic polynomial is

$$x^2 + 4y = 0.$$

We get that the plane is split into three parts

1. Above this parabola L has a pair of pairwise distinct real eigenvalues

$$\lambda_{1,2} = \frac{1}{2}\left(x \pm \sqrt{x^2 + 4y}\right)$$

2. On the parabola, L is similar to a Jordan block with real eigenvalue $\frac{x}{2}$
3. Below this parabola L has a pair of distinct complex conjugate eigenvalues

Differentially non-degenerate Nijenhuis operators

Proposition

If p is algebraically generic and differentially non-degenerate for operator field L (not necessary Nijenhuis), then:

1. L has pairwise distinct eigenvalues $\lambda_1, \dots, \lambda_k, \mu_1, \dots, \mu_s, \bar{\mu}_1, \dots, \bar{\mu}_s$. Here λ_i's are real and μ_j's and $\bar{\mu}_j$'s are pairs of complex conjugate eihenvalues
2. The differentials $\mathrm{d}\,\lambda_i, \mathrm{d}\,\mathfrak{Re}\mu_i, \mathrm{d}\,\mathfrak{Im}\mu_i$ are linearly independent at p

Proof: First, we assume that all eigenvalues are real. This implies, that

$$\chi_L(t) = (t - \lambda_1) \dots (t - \lambda_n)$$

Here some λ_i may coincide.

Differentially non-degenerate Nijenhuis operators

Locally, the following identity holds:

$$0 = \lambda_i^n - \sigma_1 \lambda_i^{n-1} - \dots - \sigma_n$$

Taking differential we get that

$$0 = \lambda_i^{n-1} \mathrm{d}\,\sigma_1 - \dots - \mathrm{d}\,\sigma_n + \chi_L'(\lambda_i) \mathrm{d}\,\lambda_i$$

If λ_i has multiplicity ≥ 2, then $\chi_L'(\lambda_i)$ vanishes and the identity yields non-trivial relation on $\mathrm{d}\,\sigma_i$. Thus, all the roots have multiplicity one. The same identity yields the system

$$\begin{pmatrix} -\chi_L'(\lambda_1)\mathrm{d}\,\lambda_1 \\ -\chi_L'(\lambda_2)\mathrm{d}\,\lambda_2 \\ \vdots \\ -\chi_L'(\lambda_n)\mathrm{d}\,\lambda_n \end{pmatrix} = \begin{pmatrix} \lambda_1^{n-1} & \lambda_1^{n-2} & \dots & 1 \\ \lambda_2^{n-1} & \lambda_2^{n-2} & \dots & 1 \\ & & \ddots & \\ \lambda_n^{n-1} & \lambda_n^{n-2} & \dots & 1 \end{pmatrix} \begin{pmatrix} \mathrm{d}\,\sigma_1 \\ \mathrm{d}\,\sigma_2 \\ \vdots \\ \mathrm{d}\,\sigma_n \end{pmatrix}$$

The matrix is Vandermonde matrix, it is non-degenerate, thus, the differentials of λ_i are non-degenerate. The proof for complex μ_i is the same.

For complex eigenvalues the calculations are the same. In the end we get that $\mathrm{d}\,\mu_i$ are linearly independent in complex sense, which implies that $\mathrm{d}\,\mathfrak{Re}\mu_i$ and $\mathrm{d}\,\mathfrak{Im}\mu_i$ are linearly independent. The proposition is proved.

Proposition (Normal form of a differentially non-degenerate operator)

Assume that Nijenhuis operator L is differentially non-degenerate at p. Then in some neighbourhood of p operator can be brought to the form:

$$
L = \left(\begin{array}{ccccc}
x^1 & 1 & 0 & \dots & 0 \\
x^2 & 0 & 1 & \dots & 0 \\
x^3 & 0 & 0 & \dots & 0 \\
 & & & \ddots & \\
x^{n-1} & 0 & 0 & \dots & 1 \\
x^n & 0 & 0 \dots & 0 &
\end{array} \right)
$$

Differentially non-degenerate Nijenhuis operators

Proof: Fix coordinates $x^1, \dots, x^n$, associated to the coefficients of characteristic polynomial of L. Recall that for coefficients of characteristic polynomial we have the following formulas (Lecture 2):

$$
\begin{aligned}
L^* \mathrm{d}\, x^i &= x^i \mathrm{d}\, x^1 + \mathrm{d}\, x^{i+1}, \quad i = 1, \dots, n-1 \\
L^* \mathrm{d}\, x^n &= x^n \mathrm{d}\, x^1.
\end{aligned}
$$

This is exactly the formula we are looking for. The Proposition is proved.

We say that p is a differentially non-degenerate singular point if p is not algebraically generic but still differentially non-degenerate. In our example in dimension two, the singular points lied on the parabola.

Proposition

Differentially non-degenerate singular points are C^2-stable. In other words, given perturbed Nijenhuis operator $\bar{L} = L + R_k$ with R_k having at point p zero of order at least two, then there exists a diffeomorphism that kills the perturbation

Differentially non-degenerate Nijenhuis operators

The proposition follows from the fact, that such perturbation does not change the differential non-degeneracy of L at point p.

Thus, we get the simplest Nijenhuis geometry with singularities

- The singularities are C^2-stable
- There exists a unique good coordinate system around singular points
- Nijenhuis operator is always $gl-$regular
- The eigenvalues, which are the good coordinates for regular chambers are at most continuous at singular points

The next question: what manifolds may carry such geometry?

Theorem

Let L be a Nijenhuis operator on a closed connected manifold M^n with a non-real eigenvalue $\lambda \in \mathbb{C} \setminus \mathbb{R}$ at least at one point. Then this number λ is an eigenvalue of L with the same algebraic multiplicity at every point of M. In short, the complex eigenvalues of Nijenhuis operator on a closed connected manifold are constants.

Proof of theorem: step 1

Define the function $\mu : \mathrm{M}^n \to \mathbb{R}$ as

$$\mu(x) = \max_{\lambda \in \operatorname{Spec} L} \mathfrak{Im} \lambda(x).$$

Here $\operatorname{Spec} L$ stands for the spectrum of L. That is at each point x we take eigenvalues of L, take their imaginary part and pick maximum of all these parts. We get, that $\mu(x) = 0$ if and only if all the eigenvalues in x are real.

The eigenvalues of an operator are continuous functions. Thus, $\mu(x)$ is continuous function on a connected compact, thus, it achieves it maximum and minimum. We are interested in the maximum and we denote the set of maximum to be $M_0 \subseteq \mathrm{M}^n$.

Proof of theorem: step 1

The set M_0 is obviously closed and, thus, compact on its own. The maximal value of μ we denote as μ_0.

Define the function $m : \mathrm{M} \to \mathbb{N}$ as

$$m(x) = \max_{\substack{\lambda \in \mathrm{Spec}\, L \\ \mathfrak{Im}\, \lambda = \mu_0}} \mathrm{mult}\, \lambda,$$

where mult stands for the multiplicity of λ. That is on the set M_0 we take only eigenvalues with imaginary part being μ_0 and take the largest multiplicity of these eigenvalues.

We denote maximum to be m_0 and the subset of M_0, where this maximum is achieved, as $\tilde{M}_0$.

Proof of theorem: step 2

The set $\tilde{M}_0$ is closed. Indeed, consider a converging series of points p_i, that lie in $\tilde{M}_0$. Denote the corresponding complex eigenvalue with real part μ_0 and multiplicity m_0 at point p_i as μ_i.

We have

$$\mu(x) = \max_{\lambda \in \mathrm{Spec}\, L} \mathfrak{Re}\lambda(x).$$

is also a continuous function on the manifold and $\mu(x)$ is bounded. Thus, we can take subseries μ_i with converging real part. We get the converging series of eigenvalues, such that limit has imaginary part μ_0.

The multiplicity of limit is m_0 as well, as the multiplicity is vanishing of some polynomial, constructed by the characteristic polynomial, using resultants. Its coefficients are the polynomials of coefficients of characteristic polynomial.

Proof of theorem: step 3

Now consider $\mathrm{p} \in \tilde{M}_0$ and denote μ to be the complex eigenvalue of multiplicity m_0. Using the Splitting theorem, we split the neighbourhood $U(\mathrm{p})$ into part with complex structure and complex coordinates $z_1, \ldots, z_{m_0}$ and $n - 2m_0$-dimensional part.

The first part has real dimension $2m_0$ and its characteristic polynomial is a product of two

$$\chi_L(t) = P(t)\bar{P}(t)$$

where $P(t) = (t + \mu)^{m_0}$ at point p. In the entire neighbourhood we have

$$P(t) = t^{m_0} + a_1 t^{m_0 - 1} + \cdots + a_{m_0}.$$

Here a_i are holomorphic. The function $f = -\mathfrak{Im} a_1$ is the sum of imaginary parts of eigenvalues. This function satisfies the condition $\Delta f = 0$ and, thus, satisfies the maximum principle. At the same time $f(\mathrm{p}) \geq f(x)$ in a given neighborhood, thus, it is constant.

Proof of theorem: step 4

The Cauchy-Riemann condition for holomorhic function a_1 imply, that the complex part is constant, then the real part is also constant and a_1 is constant on the neighbourhood.

We have that the sum of imaginary parts of eigenvalues of $P(t)$ is f and it is $m_0\mu_0$ at the same time by definition each imaginary part is $\leq \mu_0$. This is possible if and only if and only if the imaginary parts of all eigenvalues are exactly μ_0.

The roots of $P(t)$ are holomorphic almost everywhere on the $2m_0$-dimensional neighbourhood. In these good points, similar to the previous, the eigenvalues are constants. By continuity they are constants on the entire neighbourhood.

This implies that the p is contained in $\tilde{M}_0$ with some neighbourhood. Thus, $\tilde{M}_0$ is open and closed and coincides with M.

Proof of theorem: step 5

We get that μ is constant on M and everywhere has multiplicity m_0. Consider Nijenhuis operator

$$\tilde{L} = (L - \mu\,\mathrm{Id})^{m_0}(L - \bar{\mu}\,\mathrm{Id})^{m_0}.$$

It has real coefficients and at each point it has one complex eigenvalue less. Proceeding the similar way, we arrive to the operator with real eigenvalues only. Thus, the theorem is proved.

Some corollaries

Corollary

The differentially non-degenerate singular points do not appear on a closed connected manifolds.

Proof. In a neighbourhood of a differentially non-degenerate point p there always exist complex eigenvalues. They are not constant, but by the theorem, we have proved, they must be.

Corollary

Every Nijenhuis operator on the sphere S^4 has only real eigenvalues.

Proof. Assume that L at point $\mathrm{p} \in S^4$ has only complex eigenvalues. Then, they are constants and have constant multiplicity. Taking $f(L)$ we get complex structure on S^4, which has none.

Now assume that L at point $\mathrm{p} \in S^4$ has two real and two complex conjugate eigenvalues μ and $\bar{\mu}$. Complex eigenvalues have multiplicity one and are constant.

Some corollaries

They induce a two-dimensional distribution $\mathcal{D}$ on S^4

$$\mathcal{D} = \mathrm{Ker}(L - \mu)(L - \bar{\mu}).$$

Assume that S^4 is equipped with a Riemannian metric g. Consider second distribution $\mathcal{D}^\perp$.

Fix orientation form ω to be the value form of metric g. Then for vector field $\xi = \eta + \eta^\perp$ we define linear operator

$$\mathrm{J}\xi = L\eta + R\eta^\perp,$$

such that

- $R(\eta^\perp) \in \mathcal{D}^\perp$
- $g(R(\eta^\perp), R(\eta^\perp)) = g(\eta^\perp, \eta^\perp)$
- $\omega(\eta, L(\eta), \eta^\perp, R(\eta^\perp)) \geq 0$

By definition, we get $\mathrm{J}^2 = -\mathrm{Id}$, so J is an almost-complex structure.

By the classical result of Steenrod , the 4-sphere does not admit any almost complex structure.

Cyclic form of differentially non-degenerate Nijenhuis operator

The conservation law of an operator field L is function f, such that

$$\mathrm{d}\left(L^*\mathrm{d}\, f\right) = 0.$$

The name comes from the following fact. Consider a quasilinear system of PDE

$$u_t = L(u)u_x$$

where $u = (u^1, \dots, u^n)$ is a vector function of two variables x, t, $L(u)$ is a square matrix with components, depending on u. If we perform the transformation $u(\bar{u})$, we get, that $L(u)$ is transformed as operator field on the disc with coordinates $u^1, \dots, u^n$.

Assume, that f is a conservation law of L. By Poincare lemma the condition implies, that locally there exists g, such that $\mathrm{d}\, g = L^*\mathrm{d}\, f$. We have that

$$\partial_t \int\limits_{-\infty}^{\infty} f(u)\mathrm{d}\, x = \int\limits_{-\infty}^{\infty} \langle \mathrm{d}\, f, u_t\rangle \mathrm{d}\, x = \int\limits_{-\infty}^{\infty} \langle L^*\mathrm{d}\, f, u_x\rangle \mathrm{d}\, x = \int\limits_{-\infty}^{\infty} \partial_x g(u)\mathrm{d}\, x$$

Cyclic form of differentially non-degenerate Nijenhuis operator

Cyclic form of the operator field L is

$$L = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & \dots & 1 \\ \sigma_n & \sigma_{n-1} & \sigma_{n-2} & \dots & \sigma_1 \end{pmatrix},$$

where $\mathrm{d}\left(L^* \mathrm{d}\, x^n\right) = 0$. We have $\mathrm{d}\left(L^* \mathrm{d}\, x^i\right) = 0$, in other words the coordinates are conservation laws of operator field. The characteristic polynomial of the operator is

$$\chi_L(t) = t^n - \sigma_1 t^{n-1} - \dots - \sigma_n$$

The "conserrvation condition" for $\mathrm{d}\, x^n$ yields, that the differential form

$$\omega = \sigma_n \mathrm{d}\, x^1 + \dots + \sigma_1 \mathrm{d}\, x^n$$

is closed.

Cyclic form of differentially non-degenerate Nijenhuis operator

Proposition

Assume, that L is in the cyclic form. The vanishing of Nijenhuis torsion in this form is equivalent to the condition

$$\mathrm{d}\left(L^*\omega\right) = 0.$$

Proof: Consider Nijenhuis torsion as a map from one-forms to two-forms. We have

$$\begin{aligned}\mathcal{N}_L(\mathrm{d}\,x^n)(\xi,\eta) &= \mathrm{d}\,(L*\omega)(\xi,\eta) - \mathrm{d}\,\omega(L\xi,\eta) - \mathrm{d}\,\omega(\xi,L\eta) = \\ &= \mathrm{d}\,(L*\omega)(\xi,\eta).\end{aligned}$$

This completes the proof of the proposition.

Cyclic form of differentially non-degenerate Nijenhuis operator

Consider differentially non-degenerate Nijenhuis operator L. We know, that

$$\mathrm{d}\ \mathrm{tr}\, L^k = \frac{k}{k-1} L^* \mathrm{d}\ \mathrm{tr}\, L^{k-1}.$$

It is known, that $f_k = \mathrm{tr}\, L^k$ and coefficients of the characteristic polynomial σ_i are related as polynomials in the form $f_k = \sigma_k + \dots$, where $\dots$ stand for non-linear polynomial in $\sigma_{k-1}, \dots, \sigma_1$. These polynomials are called Newton-Girard polynomials. In coordinates $y^i = -\frac{1}{i}\, \mathrm{tr}\, L^i$ we have

$$\begin{aligned}
\sigma_1 &= y^1, \\
\sigma_2 &= y^2 + \frac{1}{2}(y^1)^2, \\
\sigma_3 &= y^3 + y^1 y^2 + \frac{1}{6}(y^1)^3
\end{aligned}$$

In these new coordinates differentially non-degenerate operator is in cyclic form.

The symmetries of Jordan block

Consider Nijenhuis operator L, which is similar to the Jordan block of maximal size at each point. We know, that in some coordinates $x^1, \dots, x^n$ it can be brought to constant form:

$$L = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ & & & \ddots & & \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}.$$

What are coordinate changes that preserve this matrix. We get that we are searching for coordinate change $\bar{x}(x)$, such that

$$\Big(\frac{\partial \bar{x}}{\partial x}\Big) L \Big(\frac{\partial \bar{x}}{\partial x}\Big)^{-1} = L \to \Big(\frac{\partial \bar{x}}{\partial x}\Big) L - L \Big(\frac{\partial \bar{x}}{\partial x}\Big) = 0.$$

The symmetries of Jordan block

Recall from linear algebra, that any matrix M, that commutes with L has the form

$$M = \begin{pmatrix} a_1 & a_2 & a_3 & \dots & a_{n-1} & a_n \\ 0 & a_1 & a_2 & \dots & a_{n-2} & a_{n-1} \\ & & \ddots & \ddots & \ddots & \\ 0 & 0 & 0 & \dots & a_2 & a_3 \\ 0 & 0 & 0 & \dots & a_1 & a_2 \\ 0 & 0 & 0 & \dots & 0 & a_1 \end{pmatrix}.$$

This implies, that Jacobian matrix $\frac{\partial \bar{x}}{\partial x}$ is in this form. In particular, we get that $\bar{x}^n$ depends on x^n, $\bar{x}^{n-1}$ depends on x^{n-1}, x^n and so on. The matrix is non-degenerate is and only if $a_1 \neq 0$, in our case $\frac{\partial \bar{x}^n}{\partial x^n} \neq 0$

The symmetries of Jordan block

In analytical category there is a natural formula to generate the symmetries of the Jordan block. Consider functions $f^1, \dots, f^n$, such that $(f^n)' \neq 0$. Consider matrix

$$M = x^n \operatorname{Id} + x^{n-1} L + \dots + x^1 L^{n-1}$$

and consider the following matrix-valued function

$$G(M) = f^n(M) \operatorname{Id} + f^{n-1}(M) L + \dots + f^1(M) L^{n-1}.$$

By definition $G(M)L - LG(M) = 0$, thus, it can be written in the form

$$G(M) = \begin{pmatrix} g^n & g^{n-1} & g^{n-2} & \dots & g^1 \\ 0 & g^n & g^{n-1} & \dots & g^2 \\ & & & \ddots & \\ 0 & 0 & 0 & \dots & g^n \end{pmatrix}$$

where g^i are functions of x.

The symmetries of Jordan block

Proposition

The formulas $\bar{x}^i = g^i$ define coordinate change, which is a symmetry of L.

Proof: Let us rewrite

$$G(M) = g^n \operatorname{Id} + g^{n-1} L + \cdots + g^1 L^{n-1},$$

where L is Jordan block and does not depend on x. Differentiating we get

$$\frac{\partial}{\partial x^i} G(M) = \frac{\partial M}{\partial x^i}\Big((f^n)' \operatorname{Id} + (f^{n-1})' L + \cdots + (f^1)' L^{n-1}\Big) = \\ = L^{n-i}\Big((f^n)' \operatorname{Id} + (f^{n-1})' L + \cdots + (f^1)' L^{n-1}\Big).$$

Thus, we get matrix identity

$$\frac{\partial G(M)}{\partial x^i} = L \frac{\partial G(M)}{\partial x^{i+1}}.$$

The symmetries of Jordan block

We rewrite this identity as

$$\frac{\partial G(M)}{\partial x^{n-j}} = L^j \frac{\partial G(M)}{\partial x^n}.$$

From the other hand we have

$$\begin{aligned}
\frac{\partial}{\partial x^{n-j}} G(M) &= \frac{\partial g^n}{\partial x^{n-j}} \operatorname{Id} + \frac{\partial g^{n-1}}{\partial x^{n-j}} L + \cdots + \frac{\partial g^1}{\partial x^{n-j}} L^{n-1} = \\
&= L^j \left(\frac{\partial g^n}{\partial x^n} \operatorname{Id} + \frac{\partial g^{n-1}}{\partial x^n} L + \cdots + \frac{\partial g^1}{\partial x^n} L^{n-1} \right) = \\
&= \frac{\partial g^n}{\partial x^n} L^j + \frac{\partial g^{n-1}}{\partial x^n} L^{j+1} + \cdots + \frac{\partial g^{j+1}}{\partial x^n} L^{n-1}
\end{aligned}$$

In terms of derivatives of g^i we get

$$\begin{aligned}
&\frac{\partial g^n}{\partial x^{n-j}} = \cdots = \frac{\partial g^{n-j+1}}{\partial x^{n-j}} = 0, \\
&\frac{\partial g^{n-j}}{\partial x^{n-j}} = \frac{\partial g^n}{\partial x^n}, \ldots, \frac{\partial g^1}{\partial x^{n-j}} = \frac{\partial g^{j+1}}{\partial x^n}.
\end{aligned}$$

This implies, that the Jacobian matrix is in special form, we have discussed.

The symmetries of Jordan block

In particular, it commutes with L. The non-degeneracy comes from the fact, that for $x^{n-1} = \cdots = x^1 = 0$ we have that $g^n(x^n) = f^n(x^n)$. Thus, the Jacobi matrix is non-degenerate. The proposition is proved.

Example. Consider $f^3(t) = t + t^2, f^2 = f^1 = 0$. We get

$$G(M) = \begin{pmatrix} x^3 & x^2 & x^1 \\ 0 & x^3 & x^2 \\ 0 & 0 & x^3 \end{pmatrix} + \begin{pmatrix} (x^3)^2 & 2x^2x^3 & 2x^1x^3 + (x^2)^2 \\ 0 & (x^3)^2 & 2x^2x^3 \\ 0 & 0 & (x^3)^2 \end{pmatrix}$$

The functions are

$$\begin{aligned} g^3 &= x^3 + (x^3)^2, \\ g^2 &= x^2 + 2x^2x^3, \\ g^1 &= x^1 + 2x^1x^3 + (x^2)^2. \end{aligned}$$

The coordinate change is $\bar{x}^i = g^i$.