Nijenhuis Geometry

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Lecture 9:
Homogeneous linear Nijenhuis operators and left-symmetric algebras

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Left-symmetric algebras

Consider an algebra $\mathfrak{a}$ over field $\mathbb{R}$ or $\mathbb{C}$ with operation $\star$. **The associator** of the algebra is the following trilinear operation

$$\mathcal{A}(\xi, \eta, \zeta) = \xi \star (\eta \star \zeta) - (\xi \star \eta) \star \zeta.$$

The algebra is associative if and only if its associator vanishes. The algebra is called **left-symmetric** if for arbitrary ξ, η, ζ we have

$$\mathcal{A}(\xi, \eta, \zeta) = \mathcal{A}(\eta, \xi, \zeta)$$

Obviously if the algebra is associative, then it is left-symmetric

Left-symmetric algebras

Example: Consider functions f on the line $\mathbb{R}$ with coordinate x and denote operation $\star$ as

$$f \star g = fg_x$$

Here g_x stands for derivative of g in x. The associator yields

$$\begin{aligned} & f \star (g \star h) - (f \star g) \star h = f \star (gh_x) - (fg_x) \star h = \\ & = fg_x h_x + fgh_{xx} - fg_x h_x = fgh_{xx} \end{aligned}$$

Obviously, the algebra is not associative, but it is left-symmetric

Example: Consider manifold M^n with flat symmetric connection ∇. Define the operation on vector fields as $\xi \star \eta = \nabla_\xi \eta$. The associator takes the form

$$\mathcal{A}(\xi, \eta, \zeta) = \xi \star (\eta \star \zeta) - (\xi \star \eta) \star \zeta = \nabla_\xi \nabla_\eta \zeta - \nabla_{\nabla_\xi \eta} \zeta.$$

Left-symmetric algebras

The condition $\mathcal{A}(\xi,\eta,\zeta)=\mathcal{A}(\eta,\xi,\zeta)$ takes the form

$$\nabla_\xi\nabla_\eta\zeta-\nabla_{\nabla_\xi\eta}\zeta-\nabla_\eta\nabla_\xi\zeta+\nabla_{\nabla_\eta\xi}\zeta=\nabla_\xi\nabla_\eta\zeta-\nabla_\eta\nabla_\xi\zeta-\nabla_{(\nabla_\eta\xi-\nabla_\xi\eta)}\zeta=$$
$$=\nabla_\xi\nabla_\eta\zeta-\nabla_\eta\nabla_\xi\zeta-\nabla_{[\xi,\eta]}\zeta=0.$$

Here we used both the symmetry condition for connection and vanishing of the curvature tensor.

The main property of left-symmetric algebras is as follows (after that they got their alternative name pre-Lie algebras)

Proposition

Consider a left-symmetric algebra $\mathfrak{a}$ with operation $\star$. Then the operation

$$[\xi,\eta]=\xi\star\eta-\eta\star\xi$$

defines Lie algebra structure.

Proof of proposition

The operation is bilinear and skew-symmetric, thus, the only thing we need to prove is the Jacobi condition. For arbitrary triple $\xi, \eta, \zeta \in \mathfrak{a}$ we have

$$\begin{aligned}
&[\xi,[\eta,\zeta]] + [\eta,[\zeta,\xi]] + [\zeta,[\xi,\eta]] = \\
=&[\xi, \eta\star\zeta - \zeta\star\eta] + [\eta, \zeta\star\xi - \xi\star\zeta] + [\zeta, \xi\star\eta - \eta\star\xi] = \\
=&\xi\star(\eta\star\zeta) - \xi\star(\zeta\star\eta) - (\eta\star\zeta)\star\xi + (\zeta\star\eta)\star\xi + \\
+&\eta\star(\zeta\star\xi) - \eta\star(\xi\star\zeta) - (\zeta\star\xi)\star\eta + (\xi\star\zeta)\star\eta + \\
+&\zeta\star(\xi\star\eta) - \zeta\star(\eta\star\xi) - (\xi\star\eta)\star\zeta + (\eta\star\xi)\star\zeta = \\
=&\mathcal{A}(\xi,\eta,\zeta) - \mathcal{A}(\xi,\zeta,\eta) + \mathcal{A}(\eta,\zeta,\xi) - \mathcal{A}(\eta,\xi,\zeta) + \\
+&\mathcal{A}(\zeta,\xi,\eta) - \mathcal{A}(\zeta,\eta,\xi)
\end{aligned}$$

We see, that the Jacobi condition is an alternated sum of associators (this holds for arbitrary algebra). Thus, the left-symmetry (as well as right-symmetry or symmetry in first and third argument) yields zero: the corresponding terms cancel out.

Left-symmetric algebras

Denote $L_\xi \eta = \xi \star \eta$ and $R_\eta \xi = \xi \star \eta$. Both L, R are not representations: $\mathfrak{a}$ is not associative. Yet, L_ξ is a representtaion of the Lie algebra, associated with $\mathfrak{a}$.

Indeed, the left-symmetry condition has the form

$$\xi \star (\eta \star \zeta) - (\xi \star \eta) \star \zeta - \eta \star (\xi \star \zeta) + (\eta \star \xi) \star \zeta = 0$$

It can be rewritten as

$$L_\xi L_\eta \zeta - L_\eta L_\xi \zeta - L_{[\xi,\eta]} \zeta = 0,$$

where $[\xi, \eta] = \xi \star \eta - \eta \star \xi$. The corresponding representation is called regular representation of the Lie algebra, associated with $\mathfrak{a}$. In case of R we have

$$R_{\xi \star \eta} - R_\eta R_\xi = L_\xi R_\eta - R_\eta L_\xi$$

which does not seem to carry any interesting information on its own.

Left-symmetric algebras

If $\mathfrak{a}$ is a finite dimensional algebra, then it has a natural structure of a smooth $n-$dimensional affine manifold.

Consider a point η of this manifold. Tangent space to η is naturally identified with $\mathfrak{a}$ itself. Thus, one can define a tensor field R of type $(1,1)$ as follows: R acts on element ξ from tangent space at point η as

$$R_\eta \xi = \xi \star \eta.$$

Thus, we interpret R not as map from $\mathfrak{a} \to \mathfrak{gl}(n)$ **but as operator field on $\mathfrak{a}$.**.

Theorem (Winterhalder)

Finite dimensional algebra $\mathfrak{a}$ over $\mathbb{R}$ or $\mathbb{C}$ is left-symmetric if and only if associated operator field R is Nijenhuis.

Proof of theorem

Fix basis $\eta_1, \dots, \eta_n$ and denote the structure constants of $\mathfrak{a}$ to be a_{ij}^k. Denote the corresponding coordinates as $x^1, \dots, x^n$. The operator field R has entries

$$R_i^k = a_{ij}^k x^j.$$

We have

$$\frac{\partial}{\partial x^r}\left(\mathcal{N}_R\right) = \frac{\partial}{\partial x^r}\left(\frac{\partial R_i^\alpha}{\partial x^j} R_\alpha^k - \frac{\partial R_j^\alpha}{\partial x^i} R_\alpha^k - \frac{\partial R_i^k}{\partial x^\alpha} R_j^\alpha + \frac{\partial R_j^k}{\partial x^\alpha} R_i^\alpha\right) =$$
$$= a_{ij}^\alpha a_{\alpha r}^k - a_{ji}^\alpha a_{\alpha r}^k - a_{i\alpha}^k a_{jr}^\alpha + a_{j\alpha}^k a_{ir}^\alpha.$$

the last equation can be written as

$$(\eta_i \star \eta_j) \star \eta_r - (\eta_j \star \eta_i) \star \eta_r - \eta_i \star (\eta_j \star \eta_r) + \eta_j \star (\eta_i \star \eta_r) =$$
$$= \mathcal{A}(\eta_i, \eta_j, \eta_r) - \mathcal{A}(\eta_j, \eta_i, \eta_r).$$

Thus, if $\mathfrak{a}$ is left-symmetric, then all the derivatives of $\mathcal{N}_R$ vanish and it has constant entries. As $R = 0$ at the origin, then $\mathcal{N}_R = 0$. On the other hand if $\mathcal{N}_R = 0$, then the algebra is left symmetric.

Linear operator fields

As we have noticed, the components of R are homogeneous linear polynomials. From the previous, we get, that if components of R depend linearly on coordinates, then locally the neighbourhood of p has a structure of left-symmetric algebra. Thus, we have established an important relation

$$\left[\begin{array}{c}\text{Left-symmetric}\\ \text{algebras}\end{array}\right] = \left[\begin{array}{c}\text{Nijenuhis operators on affine spaces}\\ \text{with linear homogeneous entries}\end{array}\right]$$

An isomorphism of the algebras becomes a diffeomorphism of affine manifolds with fixed coordinate origins.

Besides R one may also define operator field L. It is not Nijenhuis, but it provides helpfull geometric structure in distinguishing the algebras.

Linear bivectors in Poisson geometry

One might recall that in the case of finite-dimensional Lie algebra $\mathfrak{g}$ over $\mathbb{R}$ or $\mathbb{C}$ the similar relation exists.

Taking the natural affine structure on dual space $\mathfrak{g}^*$ we can define bivector $\mathcal{P}$ as follows: at point $x \in \mathfrak{g}^*$ the bivector $\mathcal{P}_x$ acts on a pair $\xi, \eta \in T_x^*\mathfrak{g}^*$ as

$$\mathcal{P}_x(\xi, \eta) = x([\xi, \eta]).$$

Here we identify $T_x^*\mathfrak{g}^*$ with $\mathfrak{g}^{**} = \mathfrak{g}$. Thus, linear homogeneous Poisson brackets are the same as Lie-Poisson brackets on dual spaces of Lie algebras.

Classification theorem in dimensions one and two

Example: Consider arbitrary one-dimensional algebra $\mathfrak{a}$. In basis η we have $\eta \star \eta = a\eta$.

1. If $a = 0$ then we have a trivial algebra
2. If $a \neq 0$, then taking $\bar{\eta} = \frac{1}{a}\eta$ yields structure relation $\bar{\eta} \star \bar{\eta} = \frac{1}{a^2} a\eta = \bar{\eta}$

We see that in dimension one, all algebras are commutative and associative, thus, in particular, they are left-symmetric.

Now consider a two-dimensional Lie algebra $\mathfrak{g}$. We know that up to an isomorphism there are exactly two such algebras: commutative and one non-commutative with structure relation $[\eta_1, \eta_2] = \eta_1$.

Classification theorem in dim $= 2$

Theorem

Up to isomorphism there are two continuous families and 10 exceptional two dimensional real left-symmetric algebras. The complete list of normal forms is presented in Table 1 and Table 2 below. For every algebra we give

- *All non-zero structure relations for a given basis* η_1, η_2
- *Right-adjoint operator of* $\mathfrak{a}$ *in coordinates* x, y*, associated with basis* η_1, η_2*. We denote it as R*
- *Left-adjoint operator of* $\mathfrak{a}$ *in coordinates* x, y*, associated with basis* η_1, η_2*. We denote it as L.*

The letter $\mathfrak{b}$ *stands for algebras with non-abelian associated Lie algebra and* $\mathfrak{c}$ *for algebras with Abelian associated Lie algebra.*

Classification theorem: table 1

Name	Structure relations	L	R
$\mathfrak{b}_{1,\alpha}$	$\eta_2 \star \eta_1 = \eta_1,$ $\eta_2 \star \eta_2 = \alpha\eta_2$	$\begin{pmatrix} y & 0 \\ 0 & \alpha y \end{pmatrix}$	$\begin{pmatrix} 0 & x \\ 0 & \alpha y \end{pmatrix}$
$\mathfrak{b}_{2,\beta}, \beta \neq 1$	$\eta_1 \star \eta_2 = \eta_1,$ $\eta_2 \star \eta_1 = \beta\eta_1$ $\eta_2 \star \eta_2 = \eta_2$	$\begin{pmatrix} \beta y & x \\ 0 & y \end{pmatrix}$	$\begin{pmatrix} y & \beta x \\ 0 & y \end{pmatrix}$
$\mathfrak{b}_3$	$\eta_2 \star \eta_1 = \eta_1,$ $\eta_2 \star \eta_2 = \eta_1 + \eta_2$	$\begin{pmatrix} y & y \\ 0 & y \end{pmatrix}$	$\begin{pmatrix} 0 & x+y \\ 0 & y \end{pmatrix}$
$\mathfrak{b}_4^+$	$\eta_1 \star \eta_1 = \eta_2,$ $\eta_2 \star \eta_1 = -\eta_1$ $\eta_2 \star \eta_2 = -2\eta_2$	$\begin{pmatrix} -y & 0 \\ x & -2y \end{pmatrix}$	$\begin{pmatrix} 0 & -x \\ x & -2y \end{pmatrix}$
$\mathfrak{b}_4^-$	$\eta_1 \star \eta_1 = -\eta_2,$ $\eta_2 \star \eta_1 = -\eta_1$ $\eta_2 \star \eta_2 = -2\eta_2$	$\begin{pmatrix} -y & 0 \\ -x & -2y \end{pmatrix}$	$\begin{pmatrix} 0 & -x \\ -x & -2y \end{pmatrix}$
$\mathfrak{b}_5$	$\eta_1 \star \eta_2 = \eta_1,$ $\eta_2 \star \eta_2 = \eta_1 + \eta_2$	$\begin{pmatrix} 0 & x+y \\ 0 & y \end{pmatrix}$	$\begin{pmatrix} y & y \\ 0 & y \end{pmatrix}$

Classification theorem: table 2

Name	Structure relations	$L = R$
$\mathfrak{c}_1$		$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$
$\mathfrak{c}_2$	$\eta_2 \star \eta_2 = \eta_2$	$\begin{pmatrix} 0 & 0 \\ 0 & y \end{pmatrix}$
$\mathfrak{c}_3$	$\eta_2 \star \eta_2 = \eta_1$	$\begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix}$
$\mathfrak{c}_4$	$\eta_2 \star \eta_2 = \eta_2$ $\eta_2 \star \eta_1 = \eta_1$ $\eta_1 \star \eta_2 = \eta_1$	$\begin{pmatrix} y & x \\ 0 & y \end{pmatrix}$
$\mathfrak{c}_5^+$	$\eta_2 \star \eta_2 = \eta_2$ $\eta_2 \star \eta_1 = \eta_1$ $\eta_1 \star \eta_2 = \eta_1$ $\eta_1 \star \eta_1 = \eta_2$	$\begin{pmatrix} y & x \\ x & y \end{pmatrix}$
$\mathfrak{c}_5^-$	$\eta_2 \star \eta_2 = \eta_2$ $\eta_2 \star \eta_1 = \eta_1$ $\eta_1 \star \eta_2 = \eta_1$ $\eta_1 \star \eta_1 = -\eta_2$	$\begin{pmatrix} y & x \\ -x & y \end{pmatrix}$

Proof of Classification theorem: step 1

First, we check that algebras in the Tables 1 and 2 are indeed left-symmetric. For algebras $\mathfrak{b}_{1,\alpha}, \mathfrak{b}_{2,\beta}, \mathfrak{b}_3, \mathfrak{b}_5$, $\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{c}_3$ and $\mathfrak{c}_4$ their right-adjoint operators are Nijenhuis by results, proved in Lecture 4. For the rest we check directly.

For $\mathfrak{b}_4^+$ we have $\det R = x^2, \operatorname{tr} R = -2y$ and:

$$\begin{pmatrix} 2x, & 0 \end{pmatrix} = \begin{pmatrix} 0, & -2 \end{pmatrix} \begin{pmatrix} -2y & x \\ -x & 0 \end{pmatrix}$$

For b_4^- we have $\det R = -x^2, \operatorname{tr} R = -2y$ and:

$$\begin{pmatrix} -2x, & 0 \end{pmatrix} = \begin{pmatrix} 0, & -2 \end{pmatrix} \begin{pmatrix} -2y & x \\ x & 0 \end{pmatrix}$$

Proof of Classification theorem: step 1

For $\mathfrak{c}_5^+$ we have $\det R = y^2 - x^2, \operatorname{tr} R = 2y$ and:

$$\begin{pmatrix} -2x, & 2y \end{pmatrix} = \begin{pmatrix} 0, & 2 \end{pmatrix} \begin{pmatrix} y & -x \\ -x & y \end{pmatrix}$$

For $\mathfrak{c}_5^-$ we have $\det R = y^2 + x^2, \operatorname{tr} R = 2y$ and:

$$\begin{pmatrix} 2x, & 2y \end{pmatrix} = \begin{pmatrix} 0, & 2 \end{pmatrix} \begin{pmatrix} y & -x \\ x & y \end{pmatrix}$$

As right-adjoint operators are Nijenhuis, then the algebras are left-symmetric.

Proof of Classification theorem: step 2

Algebras from Table 1 and Table 2 are not isomorphic, as their associated Lie algebras are not isomorphic to each other. First, we show that algebras from Table 1 are not isomorphic

- Consider $F = \det L$. This function is identically zero on $\mathfrak{b}_5, \mathfrak{b}_{1,0}, \mathfrak{b}_{2,0}$ only, thus, these algebras are not isomorphic to others from Table 1. R for $\mathfrak{b}_{1,0}$ has zero eigenvalues everywhere, R for $\mathfrak{b}_{2,0}$ consists of semisimple elements, R for $\mathfrak{b}_5$ is Jordan block with non-zero eigenvalues almost everywhere.
- Consider $F = \det R$. It is identically zero for $\mathfrak{b}_{1,\alpha}$ and $\mathfrak{b}_3$. Thus, they are not isomorphic to the rest of the algebras in Table 1.
- Operator field L for $\mathfrak{b}_{1,\alpha}$ is diagonalizable, while L for $\mathfrak{b}_3$ are Jordan blocks almost everywhere. Thus, these algebras are not isomorphic.
- Fix α_0 and consider $F = \alpha_0 \det L - (\operatorname{tr} R)^2$. This function vanishes only for $\mathfrak{b}_{1,\alpha_0}$ and non-zero for $\alpha \neq \alpha_0$. Thus, algebras $\mathfrak{b}_{1,\alpha}$ for different values of parameter α are not isomorphic.

Proof of Classification theorem: step 2

- Consider $F = (\operatorname{tr} R)^2 - 4 \det R$ (that is the discriminant of the characteristic polynomial of R). It vanishes identically for $\mathfrak{b}_{2,\beta}$ and not for $\mathfrak{b}_4^+, \mathfrak{b}_4^-$. Thus, $\mathfrak{b}_{2,\beta}$ is not isomorphic to the rest of the algebras from Table 1.
- Fix $\beta_0 \neq 1$ and consider $F = \det L - \left(1 - \frac{1}{\beta_0}\right) \det R$. It vanishes for $\mathfrak{b}_{2,\beta_0}$ and does not vanish for $\mathfrak{b}_{2,\beta}$ for $\beta \neq \beta_0$. Thus, the algebras $\mathfrak{b}_{2,\beta}$ are not isomorphic for different values of the parameters.
- Denote the function

$$T(x) = \begin{cases} 0 & x \geq 0, \\ 1 & x < 0. \end{cases}$$

  Take $F = T((\operatorname{tr} R)^2 - 4 \det R)$. It vanishes for $\mathfrak{b}_4^-$ and does not vanish for $\mathfrak{b}_4^+$.

Proof of Classification theorem: step 3

Now we do the same for Table 2:

- For algebra $\mathfrak{c}_1$ the operator field $R = L$ is zero, but it is not zero for the rest of the Table. Thus, this algebra is not isomorphic to any algebra from the rest of Table 2.
- Take $F = \operatorname{tr} R$. It vanishes for $\mathfrak{c}_3$, but not for $\mathfrak{c}_2, \mathfrak{c}_4, \mathfrak{c}_5^+, \mathfrak{c}_5^-$. Thus, $\mathfrak{c}_3$ is not isomorphic to $\mathfrak{c}_2, \mathfrak{c}_4, \mathfrak{c}_5^+, \mathfrak{c}_5^-$.
- Take $F = \det R$. It vanishes for $\mathfrak{c}_2$, but not for $\mathfrak{c}_4, \mathfrak{c}_5^+, \mathfrak{c}_5^-$. Thus, the $\mathfrak{c}_2$ is not isomorphic to $\mathfrak{c}_4, \mathfrak{c}_5^+, \mathfrak{c}_5^-$.
- Take $F = T(\det R)$. It is zero for $\mathfrak{c}_5^-$ and nonzero for $\mathfrak{c}_5^+$ and $\mathfrak{c}_4$. Thus, it is not isomorphic to $\mathfrak{c}_5^+, \mathfrak{c}_4$.
- Notice, that R for $\mathfrak{c}_5^+$ almost everywhere has pairwise distinct eigenvalues, while $\mathfrak{c}_4$ is Jordan block everywhere.

Proof of Classification theorem: step 4

Lemma

Let $R, Q \in \mathfrak{gl}(n, \mathbb{R})$ and $[R, Q] = \lambda Q$ for $\lambda \neq 0$. Then Q is nilpotent.

Proof: An operator $\mathrm{ad}_R : \mathfrak{gl}(n, \mathbb{R}) \to \mathfrak{gl}(n, \mathbb{R})$ is defined by the formula $\mathrm{ad}_R Q = [R, Q] = \lambda Q$. From the properties of the matrix commutator it follows that $\mathrm{ad}_R Q^n = n\lambda Q^n$.

Suppose now that $Q^n \neq 0$ for all $n \in \mathbb{N}$. It means that the finite dimensional operator ad_R has an infinite set of eigenvectors.

Lemma

Every two-dimensional commutative subalgebra $\mathfrak{h} \subset \mathfrak{gl}(2, \mathbb{R})$ contains the one-dimensional subspace spanned by the identity matrix.

Proof: By Schur's theorem the maximal dimension of commutative subalgebra in $\mathfrak{gl}(2, \mathbb{R})$ is two. This implies the statement of the Lemma.

Denote N to be the dimension of the image of the regular representation of $\mathfrak{a}$.

Proof of Classification theorem: $N = 0$

This implies that $L_\eta = 0$ for an arbitrary $\eta \in \mathfrak{a}$. This is $\mathfrak{c}_1$.

Proof of Classification theorem: $N = 1, \gamma = 0$

For $N = 1$ we may assume that in basis ξ_1, ξ_2 the element ξ_1 spans the kernel of map L, that is $L_{\xi_1} = 0$. We get $L_{[\xi_1,\xi_2]} = [L_{\xi_1}, L_{\xi_2}] = 0$ and

$$[\xi_1, \xi_2] = \gamma\xi_1. \tag{1}$$

If $\gamma = 0$ in (1) we get that $[\xi_1, \xi_2] = 0$. Thus, in given basis structure constants for $\mathfrak{a}$ are

$$\begin{aligned} &\xi_1 \star \xi_1 = \xi_1 \star \xi_2 = \xi_2 \star \xi_1 = 0, \\ &\xi_2 \star \xi_2 = b\xi_1 + a\xi_2, \end{aligned} \tag{2}$$

where a, b some constants.

Proof of Classification theorem: $N = 1, \gamma = 0$

If $a \neq 0$ in (2), consider the change of basis $\eta_1 = \xi_1$ and $\eta_2 = \frac{b}{a^2}\xi_1 + \frac{1}{a}\xi_2$. The structure relations (2) take form

$$\eta_1 \star \eta_1 = \eta_1 \star \eta_2 = \eta_2 \star \eta_1 = 0,$$
$$\eta_2 \star \eta_2 = \frac{1}{a^2}\xi_2 \star \xi_2 = \left(\frac{b}{a^2}\xi_1 + \frac{1}{a}\xi_2\right) = \eta_2.$$

This is $\mathfrak{c}_2$.

If $a = 0$ in (2), then $b \neq 0$ (otherwise $\dim \operatorname{Im} L = 0$). After the change of basis $\eta_1 = b\xi_1, \eta_2 = \xi_2$ relations (2) take form

$$\eta_1 \star \eta_1 = \eta_1 \star \eta_2 = \eta_2 \star \eta_1 = 0,$$
$$\eta_2 \star \eta_2 = b\xi_1 = \eta_1.$$

This is $\mathfrak{c}_3$.

Proof of Classification theorem: $N = 1, \gamma \neq 0$

If $\gamma \neq 0$, then changing coordinates $\xi_1' = \xi_1$ and $\xi_2' = -\frac{1}{\gamma}\xi_2$ yields $[\xi_1', \xi_2'] = -\xi_1'$. Thus, we may assume further that $\gamma = -1$. We get $\xi_1 \star \xi_2 - \xi_2 \star \xi_1 = \xi_1$.

The structure relations in this case are

$$\begin{aligned} \xi_1 \star \xi_1 &= \xi_1 \star \xi_2 = 0, \\ \xi_2 \star \xi_1 &= \xi_1, \\ \xi_2 \star \xi_2 &= b\xi_1 + a\xi_2, \end{aligned} \tag{3}$$

for some constants a and b.

Proof of Classification theorem: $N = 1, \gamma \neq 0$

If $a \neq 1$ in (3), then after the change of basis

$$\eta_1 = \xi_1, \eta_2 = -\frac{b}{1-a}\xi_1 + \xi_2$$

the relations (3) take form

$$\eta_1 \star \eta_1 = 0$$
$$\eta_1 \star \eta_2 = 0,$$
$$\eta_2 \star \eta_1 = \Big(-\frac{b}{1-a}\xi_1 + \xi_2\Big) \star \xi_1 = \xi_1 = \eta_1,$$
$$\eta_2 \star \eta_2 = \Big(-\frac{b}{1-a}\xi_1 + \xi_2\Big) \star \Big(-\frac{b}{1-a}\xi_1 + \xi_2\Big) = -\frac{b}{1-a}\xi_1 + b\xi_1 + a\xi_2 =$$
$$= a\Big(-\frac{b}{1-a}\xi_1 + \xi_2\Big) = a\eta_2.$$

Renaming a as α, we get $\mathfrak{b}_{1,\alpha}$.

Proof of Classification theorem: $N = 1, \gamma \neq 0$

If in (3) constants $a = 1$ and $b = 0$, then we get

$$\begin{aligned} \xi_1 \star \xi_1 &= \xi_1 \star \xi_2 = 0, \\ \xi_2 \star \xi_1 &= \xi_1, \\ \xi_2 \star \xi_2 &= \xi_2, \end{aligned}$$

This is $\mathfrak{b}_{1,1}$.

If $a = 1$ and $b \neq 0$, then change of basis $\eta_1 = b\xi_1, \eta_2 = \xi_2$ yields

$$\begin{aligned} \eta_1 \star \eta_1 &= \eta_1 \star \eta_2 = 0, \\ \eta_2 \star \eta_1 &= b\xi_2 \star \xi_1 = b\xi_1 = \eta_1, \\ \eta_2 \star \eta_2 &= \xi_2 \star \xi_2 = b\xi_1 + \xi_2 = \eta_1 + \eta_2. \end{aligned}$$

This is $\mathfrak{b}_5$.

Proof of Classification theorem: $N = 2$ case $\mathfrak{c}$

L defines a faithful representation of the associates Lie algebra for $\mathfrak{a}$. By Lemma image contains identity. W.l.o.g. we may assume that in given basis ξ_1, ξ_2

$$L_{\xi_2} = \mathrm{Id}\,.$$

This yields

$$L_{[\xi_1,\xi_2]} = [L_{\xi_1}, L_{\xi_2}] = [L_{\xi_1}, \mathrm{Id}] = 0$$

And, as representation is faithful, we have $[\xi_1, \xi_2] = 0$. The structure relations in this case are:

$$\begin{aligned} \xi_2 \star \xi_1 &= \xi_1 \star \xi_2 = \xi_1, \\ \xi_2 \star \xi_2 &= \xi_2, \\ \xi_1 \star \xi_1 &= a\xi_1 + b\xi_2, \end{aligned} \tag{4}$$

for some constants a and b.

Proof of Classification theorem: $N = 2$ case $\mathfrak{c}$

Assume that $\frac{a^2}{4} + b = 0$. Then after change of basis

$$\eta_1 = \xi_1 - \frac{a}{2}\xi_2,$$
$$\eta_2 = \xi_2$$

we get

$$\eta_2 \star \eta_1 = \eta_1,$$
$$\eta_1 \star \eta_2 = \left(\xi_1 - \frac{a}{2}\xi_2\right) \star \xi_2 = \xi_1 - \frac{a}{2}\xi_2 = \eta_1,$$
$$\eta_2 \star \eta_2 = \eta_2,$$
$$\eta_1 \star \eta_1 = \left(\xi_1 - \frac{a}{2}\xi_2\right) \star \left(\xi_1 - \frac{a}{2}\xi_2\right) = a\xi_1 + b\xi_2 - a\xi_1 + \frac{a^2}{4}\xi_2 = 0.$$

This is $\mathfrak{c}_4$.

Proof of Classification theorem: $N = 2$ case $\mathfrak{c}$

If $\frac{a^2}{4} + b \neq 0$, then the change of basis

$$\eta_1 = \frac{1}{\sqrt{|\frac{a^2}{4} + b|}}\xi_1 - \frac{a}{2\sqrt{|\frac{a^2}{4} + b|}}\xi_2,$$
$$\eta_2 = \xi_2.$$

yields

$$\eta_2 \star \eta_1 = \eta_1, \quad \eta_1 \star \eta_2 = \eta_1, \quad \eta_2 \star \eta_2 = \eta_2,$$
$$\eta_1 \star \eta_1 =$$
$$= \Big(\frac{1}{\sqrt{|\frac{a^2}{4} + b|}}\xi_1 - \frac{a}{2\sqrt{|\frac{a^2}{4} + b|}}\xi_2\Big) \star \Big(\frac{1}{\sqrt{|\frac{a^2}{4} + b|}}\xi_1 - \frac{a}{2\sqrt{|\frac{a^2}{4} + b|}}\xi_2\Big) =$$
$$= \frac{\frac{a^2}{4} + b}{|\frac{a^2}{4} + b|}\eta_2.$$

Depending on the sign of $\frac{a^2}{4} + b$ we get either structure relations for c_5^+ or c_5^-.

Proof of Classification theorem: $N = 2$ case $\mathfrak{b}$

For appropriate basis ξ_1, ξ_2 we have $[L_{\xi_1}, L_{\xi_2}] = L_{\xi_1}$ and $[\xi_1, \xi_2] = \xi_1$. By lemma we proved earlier L_{ξ_1} is nilpotent and not zero.

Assume that $\operatorname{Im} L_{\xi_1}$ is spanned by ξ_1. This yields

$$\xi_1 \star \xi_1 = 0, \quad \xi_1 \star \xi_2 = \gamma \xi_1,$$
$$\xi_2 \star \xi_1 = (\gamma - 1)\xi_1,$$
$$\xi_2 \star \xi_2 = a\xi_1 + b\xi_2,$$

where $\gamma \neq 0$. The left-symmetry of associator yields:

$$\begin{aligned}
0 =& \mathcal{A}(\xi_1, \xi_2, \xi_2) - \mathcal{A}(\xi_2, \xi_1, \xi_2) = \\
=& (\xi_1 \star \xi_2) \star \xi_2 - \xi_1 \star (\xi_2 \star \xi_2) - (\xi_2 \star \xi_1) \star \xi_2 + \xi_2 \star (\xi_1 \star \xi_2) = \\
=& \gamma^2 \xi_1 - b\gamma\xi_1 - \gamma(\gamma - 1)\xi_1 + \gamma(\gamma - 1)\xi_1 = \\
=& \gamma(b - \gamma)\xi_1.
\end{aligned}$$

Proof of Classification theorem: $N = 2$ case $\mathfrak{b}$

We get the relations

$$\begin{aligned} &\xi_1 \star \xi_1 = 0, \quad \xi_1 \star \xi_2 = \gamma \xi_1, \\ &\xi_2 \star \xi_1 = (\gamma - 1)\xi_1, \quad \xi_2 \star \xi_2 = a\xi_1 + \gamma \xi_2. \end{aligned} \tag{5}$$

If $\gamma = 1$ and $a \neq 0$ then the change of basis $\eta_1 = a\xi_1, \eta_2 = \xi_2$ yields

$$\begin{aligned} \eta_1 \star \eta_1 = 0, \quad \eta_1 \star \eta_2 = \eta_1, \\ \eta_2 \star \eta_1 = 0, \quad \eta_2 \star \eta_2 = \eta_1 + \eta_2. \end{aligned}$$

This is $\mathfrak{b}_5$.
If $\gamma = 1$ and $a = 0$ then renaming $\eta_1 = \xi_1, \eta_2 = \xi_2$ yields

$$\begin{aligned} \eta_1 \star \eta_1 = 0, \quad \eta_1 \star \eta_2 = \eta_1, \\ \eta_2 \star \eta_1 = 0, \quad \eta_2 \star \eta_2 = \eta_2. \end{aligned}$$

This is $\mathfrak{b}_{2,1}$.

Proof of Classification theorem: $N = 2$ case $\mathfrak{b}$

If $\gamma \neq 1$, then the change of basis

$$\eta_1 = \xi_1,$$
$$\eta_2 = \frac{a}{\gamma(1-\gamma)}\xi_1 + \frac{1}{\gamma}\xi_2$$

yields

$$\eta_1 \star \eta_1 = 0,$$
$$\eta_1 \star \eta_2 = \xi_1 \star \left(\frac{a}{\gamma(1-\gamma)}\xi_1 + \frac{1}{\gamma}\xi_2\right) = \xi_1 = \eta_1,$$
$$\eta_2 \star \eta_1 = \left(\frac{a}{\gamma(1-\gamma)}\xi_1 + \frac{1}{\gamma}\xi_2\right) \star \eta_1 = \frac{\gamma-1}{\gamma}\xi_1 = \left(1 - \frac{1}{\gamma}\right)\eta_1,$$
$$\eta_2 \star \eta_2 = \left(\frac{a}{\gamma(1-\gamma)}\xi_1 + \frac{1}{\gamma}\xi_2\right) \star \left(\frac{a}{\gamma(1-\gamma)}\xi_1 + \frac{1}{\gamma}\xi_2\right) =$$
$$= -\frac{a}{\gamma^2}\xi_1 + \frac{a}{\gamma(1-\gamma)}\xi_1 + \frac{a}{\gamma^2}\xi_1 + \frac{1}{\gamma}\xi_2 = \eta_2.$$

Renaming $\left(1 - \frac{1}{\gamma}\right)$ as β we get $\mathfrak{b}_{2,\beta}$ for $\beta \neq 1$.

Proof of Classification theorem: $N = 2$ case $\mathfrak{b}$

Assume now, that image of L_{ξ_1} is spanned by $a\xi_1 + \xi_2$ for some constant a.

In dimension two the image and kernel of nilpotent operator coincide. As ξ_1 and $a\xi_1 + \xi_2$ are linearly independent for all $a \in \mathbb{R}$, we have $L_{\xi_1}\xi_1 = \xi_1 \star \xi_1 = b(a\xi_1 + \xi_2)$ for some $b \neq 0$.

Consider change of basis $\eta_1 = \frac{1}{\sqrt{|b|}}\xi_1, \eta_2 = \xi_2 + a\xi_1$. have

$$L_{\eta_1}\eta_1 = \eta_1 \star \eta_1 = \frac{b}{|b|}(a\xi_1 + \xi_2) = \operatorname{sgn}(b)\,\eta_2.$$

At the same time by definition $L_{\eta_1}\eta_2 = \eta_1 \star \eta_2 = 0$ and

$$[\eta_1, \eta_2] = \frac{1}{\sqrt{|b|}}[\xi_1, \xi_2] = \frac{1}{\sqrt{|b|}}\xi_1 = \eta_1.$$

Proof of Classification theorem: $N = 2$ case $\mathfrak{b}$

The left symmetry of the associator for triple η_1, η_2, η_2 yields:

$$\begin{aligned} 0 =& \mathcal{A}(\eta_1, \eta_2, \eta_2) - \mathcal{A}(\eta_2, \eta_1, \eta_2) = \\ =& (\eta_1 \star \eta_2) \star \eta_2 - \eta_1 \star (\eta_2 \star \eta_2) - (\eta_2 \star \eta_1) \star \eta_2 + \eta_2 \star (\eta_1 \star \eta_2) = \\ =& - \eta_1 \star (\eta_2 \star \eta_2). \end{aligned}$$

This implies, that $\eta_2 \star \eta_2 = \gamma \eta_2$ for some constant γ. The left symmetry of the associator for triple η_1, η_2, η_1 yields:

$$\begin{aligned} 0 =& \mathcal{A}(\eta_1, \eta_2, \eta_1) - \mathcal{A}(\eta_2, \eta_1, \eta_1) = \\ =& (\eta_1 \star \eta_2) \star \eta_1 - \eta_1 \star (\eta_2 \star \eta_1) - (\eta_2 \star \eta_1) \star \eta_1 + \eta_2 \star (\eta_1 \star \eta_1) = \\ =& 2\eta_1 \star \eta_1 + \mathrm{sgn}(b) \eta_2 \star \eta_2 = \mathrm{sgn}(b)(2 + \gamma)\eta_2. \end{aligned}$$

As $\mathrm{sgn}(b) \neq 0$, then $\gamma = -2$.

Proof of Classification theorem: $N = 2$ case $\mathfrak{b}$

We get the structure relations

$$\eta_1 \star \eta_1 = \mathrm{sgn}(b)\,\eta_2, \quad \eta_1 \star \eta_2 = 0,$$
$$\eta_2 \star \eta_1 = -\eta_1, \quad \eta_2 \star \eta_2 = -2\eta_2.$$

For $\mathrm{sgn}(b) = 1$ these structure constants yield $\mathfrak{b}_4^+$ and for $\mathrm{sgn}(b) = -1$ they yield $\mathfrak{b}_4^-$.

Exercises

1. Prove that $\mathfrak{c}_5^+$ is isomorphic to the direct sum of two one-dimensional non-trivial algebras
2. Prove that $\mathfrak{c}_5^-$ is a real form of one-dimensional complex algebra with non-trivial multiplication