Nijenhuis Geometry

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Lecture 16:
Frolicher-Nijenhuis bracket
and Frolicher-Nijenhuis cohomology

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Some facts about exterior algebra

The differential $k-$forms on manifold M are skew-symmetric tensor fields of type $(0, k)$. The linear space of such forms we denote as Ω^k.

The exterior product of $\alpha \in \Omega^k$ and $\beta \in \Omega^m$ is an element of Ω^{k+m} that acts on a collection of vectors $\xi_1, \ldots, \xi_{k+m}$ as

$$\alpha \wedge \beta\,(\xi_1, \ldots, \xi_{k+m}) =$$
$$= \frac{1}{k!m!} \sum_{\sigma \in \mathrm{S}_{k+m}} (-1)^\sigma \alpha(\xi_{\sigma(1)}, \ldots, \xi_{\sigma(k)}) \beta(\xi_{\sigma(k+1)}, \ldots, \xi_{\sigma(k+m)}).$$

Here S_{k+m} is a group of permutation of $k + m$ elements. This operation yields a structure of algebra on

$$\Omega = \sum_{k=0}^{n} \Omega^k,$$

with Ω^0 being the space of functions on the manifold M. The corresponding algebra is called **exterior algebra**.

Some facts about exterior algebra

The properties of the exterior product are

1. $\alpha \wedge (\beta \wedge \gamma) = (\alpha \wedge \beta) \wedge \gamma$ (associativity)
2. $\alpha \wedge \beta = (-1)^{km} \beta \wedge \alpha$ (graded anticommutativity)

As an algebra Ω is generated by Ω^1 and Ω^0.

For a given vector field ξ **the interior derivative** i_ξ is

$$i_\xi \alpha(\eta_1, \ldots, \eta_{k-1}) = \alpha(\xi, \eta_1, \ldots, \eta_{k-1}).$$

Here $\eta_1, \ldots, \eta_{k-1}$ are arbitrary vector fields. We have that $i_\xi : \Omega^k \to \Omega^{k-1}$ for $k \geq 1$. We also add an extra condition $i_\xi : \Omega^0 \to 0$, for i_ξ to be defined on the entire Ω. It is derivation due to the following identity

$$i_\xi(\alpha \wedge \beta) = i_\xi \alpha \wedge \beta + (-1)^k \alpha \wedge i_\xi \beta$$

for $\alpha \in \Omega^k, \beta \in \Omega^m$.

Some facts about exterior algebra

The exterior derivative d is defined as follows:

1. For arbitrary $f \in \Omega^0$ $\mathrm{d}\, f$ is the differential of function
2. For arbitrary $f \in \Omega^0$ we have $\mathrm{d}\,(\mathrm{d}\, f) = 0$
3. $\mathrm{d}\,(\alpha \wedge \beta) = \mathrm{d}\,\alpha \wedge \beta + (-1)^k \alpha \wedge \mathrm{d}\,\beta$ for $\alpha \in \Omega^k, \beta \in \Omega^m$

The definition implies that $\mathrm{d} : \Omega^k \to \Omega^{k+1}$. The Cartan formula is

$$\mathrm{d}\,\alpha(\eta_1, \ldots, \eta_{k+1}) = \sum_{i=1}^{k+1} \mathcal{L}_{\eta_i} \left[\alpha(\eta_1, \ldots, \widehat{\eta}_i, \ldots, \eta_{k+1})\right] + \\ + \sum_{1 \le i < j \le k+1} \alpha([\eta_i, \eta_j], \ldots, \widehat{\eta}_i, \ldots, \widehat{\eta}_j, \ldots)$$

It can be written as

$$\mathcal{L}_\xi = i_\xi \mathrm{d} + \mathrm{d}\, i_\xi$$

Graded derivations of Ω: algebraic theory

The linear map $D : \Omega \to \Omega$ is called **graded derivation of degree** d if:

1. For $\alpha \in \Omega^k$ the image $D[\alpha] \in \Omega^{d+k}$
2. For $\alpha \in \Omega^k, \beta \in \Omega^m$ we have $D[\alpha \wedge \beta] = D[\alpha] \wedge \beta + (-1)^{kd} \alpha \wedge D[\beta]$

Identities imply that the graded derivation of degree d is defined by the image of Ω^0 and Ω^1.

Example: By definition we get that

- The interior derivative i_ξ is a derivation of degree -1
- The exterior derivative d is a derivation of degree 1
- The Lie derivative $\mathcal{L}_\xi$ is a derivation of degree 0

We denote the space of derivatives of degree d as $\mathfrak{Der}_d\Omega$. The space of all derivations $\mathfrak{Der}\Omega$ can be written as

$$\mathfrak{Der}\,\Omega = \bigoplus_{d=-1}^{n-1} \mathfrak{Der}_d\Omega$$

Graded derivations of Ω: algebraic theory

The linear space of derivations of a commutative ring can be equipped with a commutator. The $\mathfrak{Der}\,\Omega$ is equipped with graded commutator: for derivations D_1, D_2 of degrees d_1, d_2 we define

$$[D_1, D_2] = D_1 D_2 - (-1)^{d_1 d_2} D_2 D_1.$$

Theorem

1. *For $D_i \in \mathfrak{Der}_{d_i}\Omega$ the graded commutator $[D_1, D_2] \in \mathfrak{Der}_{d_1+d_2}\Omega$*
2. *Graded anticommutativity*

$$[D_1, D_2] = -(-1)^{d_1 d_2}[D_2, D_1]$$

3. *Graded Jacobi identity*

$$(-1)^{d_1 d_3}[D_1, [D_2, D_3]] + (-1)^{d_2 d_1}[D_2, [D_3, D_1]] + (-1)^{d_3 d_2}[D_3, [D_1, D_2]] = 0$$

Proof of theorem

The second property follows from the definition. To check the first property consider $\alpha \in \Omega^k, \beta \in \Omega^m$ and

$$D_1 D_2(\alpha \wedge \beta) = D_1\big(D_2(\alpha) \wedge \beta + (-1)^{d_2 k} \alpha \wedge D_2(\beta)\big) =$$
$$= D_1 D_2(\alpha) \wedge \beta + (-1)^{(k+d_2)d_1} D_1(\alpha) \wedge D_2(\beta) +$$
$$+ (-1)^{d_2 k} D_2(\alpha) \wedge D_1(\beta) + (-1)^{k(d_1+d_2)} \alpha \wedge D_1 D_2(\beta)$$

For the inverse order of D_1, D_2

$$D_2 D_1(\alpha \wedge \beta) = D_2 D_1(\alpha) \wedge \beta + (-1)^{(k+d_1)d_2} D_2(\alpha) \wedge D_1(\beta) +$$
$$+ (-1)^{d_1 k} D_1(\alpha) \wedge D_2(\beta) + (-1)^{k(d_1+d_2)} \alpha \wedge D_2 D_1(\beta)$$

Finally

$$D_1 D_2(\alpha \wedge \beta) - (-1)^{d_1 d_2} D_2 D_1(\alpha \wedge \beta) = [D_1, D_2](\alpha \wedge \beta) =$$
$$= [D_1, D_2](\alpha) \wedge \beta + (-1)^{k(d_1+d_2)} \alpha \wedge [D_1, D_2](\beta)$$

Thus, graded commutator is a graded derivation of degree $d_1 + d_2$.

Proof of theorem

To proof the Jacobi identity we start with

$$
\begin{aligned}
&[D_1,[D_2,D_3]] = D_1[D_2,D_3] - (-1)^{d_1(d_2+d_3)}[D_2,D_3]D_1 = \\
=&D_1\Big(D_2D_3 - (-1)^{d_2d_3}D_3D_2\Big) - (-1)^{d_1(d_2+d_3)}\Big(D_2D_3 - (-1)^{d_2d_3}D_3D_2\Big)D_1 = \\
=&D_1D_2D_3 - (-1)^{d_2d_3}D_1D_3D_2 - (-1)^{d_1(d_2+d_3)}D_2D_3D_1 + \\
+&(-1)^{d_1d_2+d_1d_3+d_2d_3}D_3D_2D_1
\end{aligned}
$$

Thus, we get

$$
\begin{aligned}
&(-1)^{d_1d_3}[D_1,[D_2,D_3]] = \\
=&(-1)^{d_1d_3}D_1D_2D_3 - (-1)^{(d_1+d_2)d_3}D_1D_3D_2 - (-1)^{d_1d_2}D_2D_3D_1 + \\
+&(-1)^{(d_1+d_3)d_2}D_3D_2D_1
\end{aligned}
$$

Proof of theorem

The entire Jacobi identity takes form

$$
\begin{aligned}
=&(-1)^{d_1 d_3} D_1 D_2 D_3 - (-1)^{(d_1+d_2)d_3} D_1 D_3 D_2 - (-1)^{d_1 d_2} D_2 D_3 D_1 + \\
+&(-1)^{(d_1+d_3)d_2} D_3 D_2 D_1 + \\
+&(-1)^{d_2 d_1} D_2 D_3 D_1 - (-1)^{(d_2+d_3)d_1} D_2 D_1 D_3 - (-1)^{d_2 d_3} D_3 D_1 D_2 + \\
+&(-1)^{(d_2+d_1)d_3} D_1 D_3 D_2 + \\
+&(-1)^{d_3 d_2} D_3 D_1 D_2 - (-1)^{(d_3+d_1)d_2} D_3 D_2 D_1 - (-1)^{d_3 d_1} D_1 D_2 D_3 + \\
+&(-1)^{(d_3+d_2)d_1} D_2 D_1 D_3 = \\
=&0
\end{aligned}
$$

The terms of the same color cancel out.

Example: As $i_\xi \in \mathfrak{Der}_{-1}\Omega$ and $\mathrm{d} \in \mathfrak{Der}_1\Omega$, then in terms of graded commutator the Cartan formula can be written as

$$\mathcal{L}_\xi = [i_\xi, \mathrm{d}]$$

Algebraic derivations

Example: Consider arbitrary derivation D of degree -1. By definition $D : \Omega^0 \to 0$. By the property of derivation we get

$$D(f \wedge \alpha) = D(f\alpha) = fD(\alpha)$$

for arbitrary $\alpha \in \Omega^k, k > 0$ and arbitrary function $f \in \Omega^0$.

By definition $D : \Omega^1 \to \Omega^0$ and this map is linear over the ring of functions. Thus, by definition this is tensor field of type $(1, 0)$, that is vector field. Thus, we have shown that $D = i_\xi$ for some ξ on Ω^0 and Ω^1. By the properties of derivation this implies that $D = i_\xi$ as operator on the entire Ω.

Note that if $i_\xi = i_\eta$, then applying to Ω^1 we get $\xi = \eta$. Moreover, for all vector fields ξ, η and all $\lambda, \mu \in \mathbb{R}$ we have

$$i_{\lambda\xi+\mu\eta} = \lambda i_\xi + \mu i_\eta.$$

Thus, the space $\mathfrak{Der}_{-1}\Omega$ is isomorphic (as a linear space) to the space of all vector fields on $\mathbb{M}$.

Algebraic derivations

We say that D of degree d is **algebraic derivation** if $D : \Omega^0 \to 0$. Similar to the example for a given D of degree d the map $\Omega^1 \to \Omega^{d+1}$ is a tensor map. Thus, for a given D of degree d there exists a tensor field K of type $(1, d+1)$ that for arbitrary $\eta_1, \ldots, \eta_{d+1}$ and arbitrary $\alpha \in \Omega^1$ we have

$$D(\alpha)(\eta_1, \ldots, \eta_{d+1}) = \alpha\Big(K(\eta_1, \ldots, \eta_{d+1})\Big)$$

We denote this operator as i_K. In this notion i_ξ is a particular case of i_K, that corresponds to $d = -1$.

Theorem

For i_K of degree d, $\alpha \in \Omega^k$ and arbitrary collection of vector fields $\eta_1, \ldots, \eta_{k+d}$ the formula holds:

$$i_K\alpha(\eta_1, \ldots, \eta_{k+d}) =$$
$$= \frac{1}{(k-1)!(d+1)!} \sum_{\sigma \in \mathrm{S}_{k+d+1}} (-1)^\sigma \alpha\Big(K(\eta_{\sigma(1)}, \ldots, \eta_{\sigma(d+1)}), \eta_{\sigma(d+2)}, \ldots, \eta_{\sigma(d+k)}\Big)$$

Proof of theorem

Consider $\beta_1, \dots, \beta_k \in \Omega^1$ and denote $\alpha = \beta_1 \wedge \dots \wedge \beta_k$. Denote also $\alpha_i = \beta_1 \wedge \dots \widehat{\beta_i}, \dots \wedge \beta_k$.

Fix collection of vectors $\eta_1, \dots, \eta_{d+1}$ and denote $\xi = K(\eta_1, \dots, \eta_{d+1})$:

$$
\begin{aligned}
i_\xi \alpha = & \, i_\xi \beta_1 \wedge \beta_2 \wedge \dots \wedge \beta_k - \beta_1 \wedge i_\xi \beta_2 \wedge \dots \wedge \beta_k + \dots + \\
& + (-1)^{k-1} \beta_1 \wedge \beta_2 \wedge \dots \wedge i_\xi \beta_k = \\
& = i_\xi \beta_1 \wedge \alpha_1 - i_\xi \beta_2 \wedge \alpha_2 + \dots + (-1)^{k-1} i_\xi \beta_k \wedge \alpha_k
\end{aligned}
$$

Substituting this into the r.h.s. of formula in theorem, we get

$$
\begin{aligned}
i_K \alpha = & \, i_K \beta_1 \wedge \alpha_1 - i_K \beta_2 \wedge \alpha_2 + \dots + (-1)^{k-1} i_K \beta_k \wedge \alpha_k = \\
& = i_K \beta_1 \wedge \beta_2 \wedge \dots \wedge \beta_k + (-1)^d \beta_1 \wedge i_K \beta_2 \wedge \dots \wedge \beta_k + \dots + \\
& + (-1)^{d(k-1)} \beta_1 \wedge \beta_2 \wedge \dots \wedge i_K \beta_k
\end{aligned}
$$

Thus, the operation is derivation and uniquely defined by its action on Ω^1 and Ω^0. On Ω^1 operation i_K yields the previously defined operation. Theorem is proved.

Vector-valued differential forms

Tensor of type $(1, k)$, skew-symmetric in lower indices is called **vector-valued differential form**. The space of such tensors we denote as Ψ^k.

Here Ψ^0 is the space of vector fields, Ψ^1 is the space of operator fields. By definition $\mathcal{N}_L \in \Psi^2$ for arbitrary $L \in \Psi^1$. The algebraic derivatives of order d are identified with Ψ^{d+1}, that is $\Psi^d \subset \mathfrak{Der}_{d-1}\Omega$.

The graded commutator on $\mathfrak{Der}\Omega$ induces operation on vector-valued differential forms. It is purely algebraic operation (it contains no derivatives of tensor fields) and is called **Richardson-Nijenhuis bracket**. The formula is

$$[K, M]_{RN} = i_K M - (-1)^{(k-1)(m-1)} i_M K$$

for $K \in \Psi^k, M \in \Psi^m$. The notion i_K stands for the inner derivation of the covariant part of tensor of type $(1, k)$. In other words, by definition

$$i_K(\xi \otimes \omega) = \xi \otimes i_K \omega.$$

Exercises on Richardson-Nijenhuis bracket

1. Prove that for $\xi \in \Psi^0, M \in \Psi^k$ the Richardson-Nijenhuis bracket is

$$[\xi, M]_{RN} = i_\xi M.$$

2. Prove that for $K, M \in \Psi^1$ the Richardson-Nijenhuis bracket coincides with commutator of operators up to a sign

$$[K, M]_{RN} = MK - KM$$

3. Consider tensor field $K \in \Psi^2$ and define operation on a pair of arbitrary vector fields ξ, η as

$$[\xi, \eta] = K(\xi, \eta).$$

The corresponding algebra structure is Lie algebra if and only if

$$[K, K]_{RN} = 0.$$

Graded derivations of Ω: differential theory

For arbitrary algebraic derivation i_K of degree d we define **Lie derivative along** K as

$$\mathcal{L}_X = [i_K, \mathrm{d}].$$

By definition $\mathcal{L}_X$ is a derivation of degree $d + 1$. For $d = -1$ we get aforementioned Cartan formula for Lie derivative $\mathcal{L}_\xi$.

Theorem

For every derivation $D \in \mathfrak{Der}_d\Omega$ there exist a unique pair of elements $K \in \Psi^d, M \in \Psi^{d+1}$, such that

$$D = \mathcal{L}_K + i_M$$

We have $M = 0$ if and only if $[D, \mathrm{d}] = 0$. The derivation is algebraic if and only if $K = 0$.

Proof: Fix d arbitrary vector fields $\eta_1, \ldots, \eta_d$ and consider map from $\Omega^0 \to \Omega^0$:

$$f \to Df(\eta_1, \ldots, \eta_d)$$

Graded derivations of Ω: differential theory

As D is derivation for $f, g \in \Omega^0$ we get

$$D(fg) = Dfg + gDf.$$

Thus, the map we defined is a derivation of ring of smooth functions. Thus, there exists vector field K, such that

$$Df(\eta_1, \ldots, \eta_d) = \mathcal{L}_{K(\eta_1, \ldots, \eta_d)} f = \mathrm{d}\, f(K(\eta_1, \ldots, \eta_d))$$

The vector field K is $C^\infty-$linear in arguments η. Thus, K is a tensor of type $(1, d)$.

The difference $D - \mathcal{L}_K$ vanishes on Ω^0 and, thus, it is algebraic. We have that there exists tensor field of type $(1, d+1)$ such that $D - \mathcal{L}_K = i_M$.

Graded commutator yields $[\mathrm{d}\,, \mathrm{d}\,] = 2\mathrm{d}\,^2 = 0$. The graded Jacobi identity in this case is

$$\begin{aligned} 0 &= (-1)^d [i_K, [\mathrm{d}\,, \mathrm{d}\,]] + (-1)^d [\mathrm{d}\,, [\mathrm{d}\,, i_K]] + [\mathrm{d}\,, [i_K, \mathrm{d}\,]] = \\ &= 2[\mathrm{d}\,, [i_K, \mathrm{d}\,]] = 2[\mathrm{d}\,, \mathcal{L}_K] \end{aligned}$$

Graded derivations of Ω: differential theory

Finally note, that map $K \to \mathcal{L}_K$ is injective, that is it has no kernel. Thus if $[\mathrm{d}\,, M] = 0$, then $M = 0$. Thus, the theorem is proved ■

Consider $L \in \Psi^k, K \in \Psi^m$ and derivations $\mathcal{L}_L, \mathcal{L}_K$. From graded Jacobi identity we get $[\mathrm{d}\,, [\mathcal{L}_L, \mathcal{L}_K]] = 0$. Thus, there exists a tensor field $(1, k+m)$ we denote as $[[L, K]]$ such, that

$$[\mathcal{L}_L, \mathcal{L}_K] = \mathcal{L}_{[[L,K]]}.$$

The corresponding tensor field is called **Frolicher-Nijenhuis bracket of L and K**. From the propertied of graded commutator we obtain:

1. $[[L, K]] = -(-1)^{kl}[[K, L]]$ for $L \in \Psi^l, K \in \Psi^k$
2. For $K \in \Psi^k, M \in \Psi^m$ and $L \in \Psi^l$

$$(-1)^{km}[[K, [[L, M]]\,]] + (-1)^{lk}[[L, [[M, K]]\,]] + (-1)^{ml}[[M, [[K, L]]\,]] = 0$$

The local formulas for Frolicher-Nijenhuis bracket

Lemma (Cartan formula)

For $K \in \Psi^k$ and $\omega \in \Omega^l$ and arbitrary $\xi_1, \ldots, \xi_{k+l}$ we have

$$
\begin{aligned}
&\mathcal{L}_K \omega(\xi_1, \ldots, \xi_{k+l}) = \\
=&\frac{1}{k!l!} \sum_{\sigma \in \mathrm{S}_{k+l}} (-1)^\sigma \mathcal{L}_{K(\xi_{\sigma(1)}, \ldots, \xi_{\sigma(k)})} \omega(\xi_{\sigma(k+1)}, \ldots, \xi_{\sigma(k+l)}) - \\
-&\frac{1}{k!(l-1)!} \sum_{\sigma \in \mathrm{S}_{k+l}} (-1)^\sigma \omega\big([K(\xi_{\sigma(1)}, \ldots, \xi_{\sigma(k)}), \xi_{\sigma(k+1)}], \xi_{\sigma(k+2)}, \ldots\big) + \\
+&\frac{(-1)^k}{2(k-1)!(l-1)!} \sum_{\sigma \in \mathrm{S}_{k+l}} (-1)^\sigma \omega\big(K([\xi_{\sigma(1)}, \xi_{\sigma(2)}], \xi_{\sigma(3)}, \ldots), \xi_{\sigma(k+2)}, \ldots\big)
\end{aligned}
$$

Proof: The formula follows from the original Cartan formula and $[i_K, \mathrm{d}] = \mathcal{L}_K$. ■

The local formulas for Frolicher-Nijenhuis bracket

For $K \in \Psi^k$ and $L \in \Psi^l$ we get the following formula for the Frolicher-Nijenhuis bracket (which we will not prove in our lectures)

$$
\begin{aligned}
&[[K, L]](\xi_1, \dots, \xi_{k+l}) = \\
&= \frac{1}{k!l!} \sum_{\sigma \in \mathrm{S}_{k+l}} (-1)^\sigma [K(\xi_{\sigma(1)}, \dots, \xi_{\sigma(k)}), L(\xi_{\sigma(k+1)}, \dots, \xi_{\sigma(k+l)})] - \\
&- \frac{1}{k!(l-1)!} \sum_{\sigma \in \mathrm{S}_{k+l}} (-1)^\sigma L\big([K(\xi_{\sigma(1)}, \dots), \xi_{\sigma(k+1)}], \xi_{\sigma(k+2)}, \dots, \xi_{\sigma(k+1)}\big) + \\
&+ \frac{(-1)^{kl}}{(k-1)!l!} \sum_{\sigma \in \mathrm{S}_{k+l}} (-1)^\sigma K\big([L(\xi_{\sigma(1)}, \dots), \xi_{\sigma(l+1)}], \xi_{\sigma(l+2)}, \dots, \xi_{\sigma(k+l)}\big) + \\
&+ \frac{(-1)^{k-1}}{2(k-1)!(l-1)!} \sum_{\sigma \in \mathrm{S}_{k+l}} (-1)^\sigma L\big(K([\xi_{\sigma(1)}, \xi_{\sigma(2)}], \dots, \xi_{\sigma(k+1)}), \xi_{\sigma(k+2)}, \dots\big) + \\
&+ \frac{(-1)^{(k-1)l}}{2(k-1)!(l-1)!} \sum_{\sigma \in \mathrm{S}_{k+l}} (-1)^\sigma K\big(L([\xi_{\sigma(1)}, \xi_{\sigma(2)}], \dots, \xi_{\sigma(l+1)}), \xi_{\sigma(l+2)}, \dots\big).
\end{aligned}
$$

The examples of calculation of Frolicher-Nijenhuis bracket

1. If $k = l = 0$ then we get that $[[K, L]]$ is just commutator of vector fields
2. Consider a case $k = 0$, that is $K = \eta$ is vector field. The formula yields

   $$[[K, L]](\xi_1, \dots, \xi_l) = [\eta, L(\xi_1, \dots, \xi_l)] - L([\eta, \xi_1], \dots, \xi_l) - \dots - L(\xi_1, \dots, [\eta, \xi$$

   This is Lie derivative of L along η.
3. For $k = 1, l = 1$ we get

   $$\begin{aligned} [[K, L]](\xi, \eta) = {} & [K\xi, L\eta] + [L\xi, K\eta] - L[K\xi, \eta] - L[\xi, K\eta] - \\ & - K[L\xi, \eta] - K[\xi, L\eta] + LK[\xi, \eta] + KL[\xi, \eta] \end{aligned}$$

   This is exactly the formula for Frolicher-Nijenhuis bracket we have used earlier
4. For $K = \mathrm{Id}$ we get that $i_K = 0$ from Cartan formula. In particular, $[[\mathrm{Id}, L]] = 0$ for arbitrary L

Exercises

1. Show that the center of $\Psi, [[\,,\,]]$ is one-dimensional and it is spanned by Id
2. Consider vector fields ξ, η and $\alpha \in \Omega^k, \beta \in \Omega^l$. For $K = \alpha \otimes \xi$ and $L = \beta \otimes \eta$ prove the formula

$$[[K, L]] = \alpha \wedge \beta \otimes [\xi, \eta] + \alpha \wedge \mathcal{L}_\xi \beta \otimes \eta - \mathcal{L}_\eta \alpha \wedge \beta \otimes \xi + \\ + (-1)^k \mathrm{d}\, \alpha \wedge i_\xi \beta \otimes \eta + (-1)^k i_\eta \alpha \wedge \mathrm{d}\, \beta \otimes \xi$$

3. Using "monster" formula calculate the components of $[[K, L]]$ in local coordinates

Frolicher-Nijenhuis cohomology

Consider algebra Ψ and assume that we have a Nijenhuis operator L. That is

$$[[L, L]] = 0.$$

One can define a following differential operator

$$\mathrm{d}_L : \Psi^k \to \Psi^{k+1},$$

which simply acts as $\mathrm{d}_L K = [[L, K]]$. The graded Jacobi identity yields

$$0 = (-1)^k [[K, [[L, L]]\,]] + (-1)^k [[L, [[L, K]]\,]] - [[L, [[K, L]]\,]] =$$
$$= 2(-1)^k [[L, [[L, K]]\,]] = \mathrm{d}_L^2 K$$

Thus, differential d_L defines cohomology groups, which we call Frolicher-Nijenhuis cohomology. We denote them as $\mathcal{H}^k_{FN}$