Nijenhuis Geometry

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Lecture 17:
Frolicher-Nijenhuis cohomologies and Frolicher-Nijenhuis torsion

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Frolicher-Nijenhuis cohomology

In previous lectures we established the following facts:

1. The space Ψ is equipped with graded Lie algebra structure, known as Frolicher-Nijenhuis bracket, which we denote as $[[\, ,\,]]$
2. The vanishing of Nijenhuis torsion for $L \in \Psi^1$ is exactly the property $[[L, L]] = 0$
3. Due to the graded Jacobi identity the operator

   $$\mathrm{d}_L : \Psi^k \to \Psi^{k+1}$$

   satisfies, for Nijenhuis operator L, condition d_L^2. Thus, the Frolicher-Nijenhuis cohomology naturally appear.
4. This construction bares similarity to the Lichnerowitz-Poisson cohomology construction for Poisson brackets

Frolicher-Nijenhuis cohomology

For $\eta \in \Psi^0$ we get
$$\mathrm{d}_L\eta(\xi) = [\eta, L\xi] - L[\eta, \xi]$$
For $K \in \Psi^1$ we get
$$\begin{aligned}\mathrm{d}_L K(\xi,\eta) = [L\xi, K\eta] + [K\xi, L\eta] - K[L\xi,\eta] - K[\xi, L\eta] - \\ - L[K\xi,\eta] - L[\xi, K\eta] + LK[\xi,\eta] + KL[\xi,\eta]\end{aligned}$$

As usual we assume that $\Psi^{-1} = 0$ and $\Psi^{n+1} = 0$. These formulas imply that

1. The group $\mathcal{H}^0_{FN}$ consists of such vector fields η, that $\mathrm{d}_L\eta = 0$. This implies, that $\mathcal{L}_\eta L = 0$, that is the flow of vector field preserves the operator field
2. The group $\mathcal{H}^1_{FN}$ is a factor of deformations of L, that is $[[L + \epsilon K, L + \epsilon K]] = O(\epsilon^2)$ by trivial defomations $\mathcal{L}_\eta L$

Nijenhuis cohomology in diagonal case: differentially non-degenerate

Consider Nijenhuis operator L in given coordinates in the form

$$L = \begin{pmatrix} x^1 & 0 & \dots & 0 \\ 0 & x^2 & \dots & 0 \\ & & \ddots & \\ 0 & 0 & \dots & x^n \end{pmatrix}$$

This is a point of general position for differentially non-degenerate operator

$$L = \begin{pmatrix} y^1 & 1 & 0 & \dots & 0 \\ y^2 & 0 & 1 & \dots & 0 \\ & & & \ddots & \\ y^{n-1} & 0 & 0 & \dots & 1 \\ y^n & 0 & 0 & \dots & 0 \end{pmatrix}$$

The calculation of cohomology for such operator is an analog of Poincare lemma for this specific Nijenhuis geometry.

Case of $\mathcal{H}^0_{FN}$

Applying the formula for Lie derivative of operator field L along η we get

$$(\mathcal{L}_\eta L)^\alpha_q = \frac{\partial L^\alpha_q}{\partial x^s}\eta^s + L^\alpha_s \frac{\partial \eta^s}{\partial x^q} - \frac{\partial \eta^\alpha}{\partial x^s} L^s_q = \\ = \delta^\alpha_q \eta^\alpha + (x^\alpha - x^q)\frac{\partial \eta^\alpha}{\partial x^q} = 0.$$

For $\alpha = q$ we get that $\eta^q = 0$. Thus, the only vector field, that preserves L is $\eta = 0$ and $\mathcal{H}^0_{FN} = 0$.

The image of d_L in Ψ^1 consists of operators

$$\begin{pmatrix} \eta^1 & (x^1 - x^2)\frac{\partial \eta^1}{\partial x^2} & \dots & (x^1 - x^n)\frac{\partial \eta^1}{\partial x^n} \\ (x^2 - x^1)\frac{\partial \eta^2}{\partial x^1} & \eta^2 & \dots & (x^2 - x^n)\frac{\partial \eta^2}{\partial x^1} \\ & & \ddots & \\ (x^n - x^1)\frac{\partial \eta^n}{\partial x^1} & (x^n - x^2)\frac{\partial \eta^n}{\partial x^2} & \dots & \eta^n \end{pmatrix}$$

Case of $\mathcal{H}^1_{FN}$

Now consider the condition $[[L, K]] = 0$. We have already performed these calculations, when we considered the problem of linearization of left-symmetric algebra, which was the direct sum of one-dimensional. Still it is a good time to repeat it:

$$
\begin{aligned}
&[[L,K]](\partial_{x^i},\partial_{x^j}) = \\
=&L[K\partial_{x^i},\partial_{x^j}] + L[\partial_{x^i},K\partial_{x^j}] + K[L\partial_{x^i},\partial_{x^j}] + K[\partial_{x^i},L\partial_{x^j}] - \\
-&[L\partial_{x^i},K\partial_{x^j}] - [K\partial_{x^i},L\partial_{x^j}] = \\
=&L[K_i^\alpha\partial_{x^\alpha},\partial_{x^j}] + L[\partial_{x^i},K_j^\alpha\partial_{x^\alpha}] + K[x^i\partial_{x^i},\partial_{x^j}] + K[\partial_{x^i},x^j\partial_{x^j}] - \\
-&[x^i\partial_{x^i},K_j^\alpha\partial_{x^\alpha}] - [K_i^\alpha\partial_{x^\alpha},x^j\partial_{x^j}] = \\
=&-\frac{\partial K_i^\alpha}{\partial x^j}x^\alpha\partial_{x^\alpha} + \frac{\partial K_j^\alpha}{\partial x^i}x^\alpha\partial_{x^\alpha} - x^i\frac{\partial K_j^\alpha}{\partial x^i}\partial_{x^\alpha} + K_j^i\partial_{x^i} + x^j\frac{\partial K_i^\alpha}{\partial x^j}\partial_{x^\alpha} - K_i^j\partial_{x^j} = \\
=&\left(K_j^i + (x^j - x^i)\frac{\partial K_i^i}{\partial x^j}\right)\partial_{x^i} + \left(K_i^j + (x^i - x^j)\frac{\partial K_j^j}{\partial x^i}\right)\partial_{x^j} + \\
+&\sum_{\alpha\neq i,j}\left((x^\alpha - x^i)\frac{\partial K_j^\alpha}{\partial x^i} - (x^\alpha - x^j)\frac{\partial K_i^\alpha}{\partial x^j}\right)\partial_\alpha
\end{aligned}
$$

Case of $\mathcal{H}^1_{FN}$

The vanishing of the bracket implies

$$K^i_j = (x^i - x^j)\frac{\partial K^i_i}{\partial x^j}$$

Define $\eta = \{\eta^1, \dots, \eta^n\}$ to be $\eta^i = K^i_i$. Thus, the above formula can be written as

$$K = \mathcal{L}_\eta L.$$

This implies, that $\mathcal{H}^1_{FN} = 0$ as well. We have a conjecture

Conjecture: All cohomologies $\mathcal{H}^k_{FN} = 0$ for $k > 1$ in the case of $L = \mathrm{diag}\{x^1, \dots, x^n\}$. This would imply that we have a kind of Poincare lemma.

Nijenhuis cohomology in diagonal case: constant eigenvalues

Consider Nijenhuis operator L in given coordinates in the form

$$L = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ & & \ddots & \\ 0 & 0 & \dots & \lambda_n \end{pmatrix},$$

where λ_i are pairwise different constants. In this case the condition $\mathcal{L}_\eta L = 0$ takes form

$$(\mathcal{L}_\eta L)^\alpha_q = \frac{\partial L^\alpha_q}{\partial x^s}\eta^s + L^\alpha_s \frac{\partial \eta^s}{\partial x^q} - \frac{\partial \eta^\alpha}{\partial x^s} L^s_q = \\ = (\lambda_\alpha - \lambda_q)\frac{\partial \eta^\alpha}{\partial x^q} = 0.$$

We get, that η^i depends only on x^i. Thus, the $\mathcal{H}^0_{FN}$ is non-trivial and infinite-dimensional: each element of the cohomology is defined by n functions of single variable.

Nijenhuis cohomology in diagonal case: constant eigenvalues

Similar to the previous case we get

$$
\begin{aligned}
&[[L,K]](\partial_{x^i},\partial_{x^j}) = \\
=&L[K\partial_{x^i},\partial_{x^j}] + L[\partial_{x^i},K\partial_{x^j}] - [L\partial_{x^i},K\partial_{x^j}] - [K\partial_{x^i},L\partial_{x^j}] = \\
=&L[K_i^\alpha\partial_{x^\alpha},\partial_{x^j}] + L[\partial_{x^i},K_j^\alpha\partial_{x^\alpha}] - [\lambda_i\partial_{x^i},K_j^\alpha\partial_{x^\alpha}] - [K_i^\alpha\partial_{x^\alpha},\lambda_j\partial_{x^j}] = \\
=&\lambda_\alpha\frac{\partial K_j^\alpha}{\partial x^i}\partial_{x^\alpha} - \lambda_\alpha\frac{\partial K_i^\alpha}{\partial x^j}\partial_{x^\alpha} - \lambda_i\frac{\partial K_j^\alpha}{\partial x^i}\partial_\alpha + \lambda_j\frac{\partial K_i^\alpha}{\partial x^j}\partial_\alpha = \\
=&(\lambda_j-\lambda_i)\frac{\partial K_j^j}{\partial x^i} - (\lambda_i-\lambda_j)\frac{\partial K_i^i}{\partial x^j} + \\
+&\sum_{\alpha\neq i,j}\left((\lambda_\alpha-\lambda_i)\frac{\partial K_j^\alpha}{\partial x^i} - (\lambda_\alpha-\lambda_j)\frac{\partial K_i^\alpha}{\partial x^j}\right)\partial_\alpha = 0.
\end{aligned}
$$

We get that each diagonal element of K depends on its own coordinate, that is K_i^i depends only on x^i.

Nijenhuis cohomology in diagonal case: constant eigenvalues

The elements that are not on the diagonal satisfy the conditions

$$(\lambda_\alpha - \lambda_i)\frac{\partial K^\alpha_j}{\partial x^i} - (\lambda_\alpha - \lambda_j)\frac{\partial K^\alpha_i}{\partial x^j} = 0 \tag{1}$$

for $\alpha \neq i, j$. The following Lemma holds.

Lemma

Fix $\alpha = 1, \dots, n$ and consider the collection of $n-1$ functions K^α_i for $i \neq \alpha$, that satisfy condition (1). Then there exists function η^α, such that $K^\alpha_i = (\lambda_\alpha - \lambda_i)\frac{\partial \eta^\alpha}{\partial x^i}$.

Proof: Consider functions

$$\bar{K}^\alpha_j(x^1, \dots, x^n) = K^\alpha_j\Big((\lambda_\alpha - \lambda_1)x^1, \dots, x^\alpha, \dots, (\lambda_\alpha - \lambda_n)x^n\Big).$$

The condition (1) takes form

$$\frac{\partial \bar{K}^\alpha_j}{\partial x^i} = \frac{\partial \bar{K}^\alpha_i}{\partial x^j}$$

Nijenhuis cohomology in diagonal case: constant eigenvalues

Thus, there exist functions $\bar{\eta}^\alpha$, such that

$$\bar{K}_i^\alpha = \frac{\partial \bar{\eta}^\alpha}{\partial x^i}.$$

Define

$$\eta^\alpha(x^1, \dots, x^n) = \bar{\eta}^\alpha\Big(\frac{1}{\lambda_\alpha - \lambda_1} x^1, \dots, x^\alpha, \dots, \frac{1}{\lambda_\alpha - \lambda_n} x^n\Big).$$

We get

$$\begin{aligned}\frac{\partial \eta^\alpha}{\partial x^i} &= \frac{1}{\lambda_\alpha - \lambda_i} \frac{\partial \bar{\eta}^\alpha}{\partial x^i}\Big(\frac{1}{\lambda_\alpha - \lambda_1} x^1, \dots, x^\alpha, \dots, \frac{1}{\lambda_\alpha - \lambda_n} x^n\Big) = \\ &= \frac{1}{\lambda_\alpha - \lambda_i} \bar{K}\Big(\frac{1}{\lambda_\alpha - \lambda_1} x^1, \dots, x^\alpha, \dots, \frac{1}{\lambda_\alpha - \lambda_n} x^n\Big) = \\ &= \frac{1}{\lambda_\alpha - \lambda_i} K_i^\alpha(x^1, \dots, x^n)\end{aligned}$$

This is exactly the statement of lemma.

Nijenhuis cohomology in diagonal case: constant eigenvalues

Using Lemma we define $\eta = \{\eta^1, \ldots, \eta^n\}$ and we get, that

$$K = M + \mathcal{L}_\eta L,$$

where M is diagonal operator with each eigenvalue depending only on corresponding coordinate.

This implies that $\mathcal{H}^1_{FN}$ is not trivial, it is infinite-dimensional and again each element is uniquely defined by n functions of single variable.

Conjecture: All cohomology $\mathcal{H}^k_{FN}$ for $k > 1$ in case of $L = \text{diag}\{\lambda_1, \ldots, \lambda_n\}$ are defined by n functions of single variables.

Nijenhuis cohomology in case of Jordan block

Assume that Nijenhuis operator L is similar to the Jordan block of maximal size with constant eigenvalue. W.l.o.g. we assume that the eigenvalue is zero. By Thompson theorem in appropriate coordinates it has the form

$$L = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

The condition $\mathcal{L}_\eta L = 0$ yields

$$(\mathcal{L}_\eta L)^\alpha_q = \frac{\partial L^\alpha_q}{\partial x^s}\eta^s + L^\alpha_s \frac{\partial \eta^s}{\partial x^q} - \frac{\partial \eta^\alpha}{\partial x^s} L^s_q = L^\alpha_s \frac{\partial \eta^s}{\partial x^q} - \frac{\partial \eta^\alpha}{\partial x^s} L^s_q = 0.$$

The latter equation implies that matrix $\frac{\partial \eta^\alpha}{\partial u^q}$ commutes with the Jordan block of maximal size.

Nijenhuis cohomology in case of Jordan block

This implies that matrix is upper triangluar with equal elements on the diagonal.

We have already performed these calculations and know, that each collection of such functions is uniquely defined by n functions of one variable: exactly the functions

$$f^{\alpha}(x^{n}) = \eta^{\alpha}(0, \dots, 0, x^{n}).$$

Thus, we get that $\mathcal{H}^0_{FN}$ for Jordan block is parametrized the same way as $\mathcal{H}^0_{FN}$ for Nijenhuis operator with constant pairwise distinct eigenvalues.

Nijenhuis cohomology in case of Jordan block

Consider $[[L, K]] = 0$. For $i, j > 1$ we get

$$[[L,K]](\partial_{x^i},\partial_{x^j}) = L[K\partial_{x^i},\partial_{x^j}] + L[\partial_{x^i},K\partial_{x^j}] - [K\partial_{x^i},L\partial_{x^j}] - [L\partial_{x^i},K\partial_{x^j}] =$$
$$= L[K_i^\alpha\partial_{x^\alpha},\partial_{x^j}] + L[\partial_{x^i},K_j^\alpha\partial_{x^\alpha}] - [K_i^\alpha\partial_{x^\alpha},\partial_{x^{j-1}}] - [\partial_{x^{i-1}},K_j^\alpha\partial_{x^\alpha}] =$$
$$= \frac{\partial K_j^{\alpha+1}}{\partial x^i}\partial_{x^\alpha} - \frac{\partial K_i^{\alpha+1}}{\partial x^j}\partial_{x^\alpha} - \frac{\partial K_j^{\alpha}}{\partial x^{i-1}}\partial_{x^\alpha} + \frac{\partial K_i^{\alpha}}{\partial x^{j-1}}\partial_{x^\alpha} = 0.$$

This yields the following system of equations

$$\frac{\partial K_j^{\alpha+1}}{\partial x^i} - \frac{\partial K_i^{\alpha+1}}{\partial x^j} - \frac{\partial K_j^{\alpha}}{\partial x^{i-1}} + \frac{\partial K_i^{\alpha}}{\partial x^{j-1}} = 0, \quad \alpha = 1, \dots, n-1,$$
$$\frac{\partial K_j^{n}}{\partial x^{i-1}} - \frac{\partial K_i^{n}}{\partial x^{j-1}} = 0,$$

The last equation implies, that there exists function η^n such, that

$$K_i^n = \frac{\partial \eta^n}{\partial x^{i-1}}.$$

Nijenhuis cohomology in case of Jordan block

Consider vector field $\eta = \{0, \dots, 0, -\eta^n\}$. Then

$$\frac{\partial \eta^\alpha}{\partial x^q} = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ & & \dots & \\ -\frac{\partial \eta^n}{\partial x^1} & -\frac{\partial \eta^n}{\partial x^2} & \dots & -\frac{\partial \eta^n}{\partial x^n} \end{pmatrix}.$$

At the same time

$$\mathcal{L}_\eta L = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ & & \dots & \\ -\frac{\partial \eta^n}{\partial x^1} & -\frac{\partial \eta^n}{\partial x^2} & \dots & -\frac{\partial \eta^n}{\partial x^n} \\ 0 & \frac{\partial \eta^n}{\partial x^1} & \dots & \frac{\partial \eta^n}{\partial x^{n-1}} \end{pmatrix}$$

Nijenhuis cohomology in case of Jordan block

Now consider $\bar{K} = K - \mathcal{L}_\eta L$. By definition we have

$$[[L, \bar{K}]] = [[L, K]] - [[L, \mathcal{L}_\eta L]] = 0.$$

At the same time the last row of the $\bar{K}$ is zero, except, probably, $\bar{K}^n_1$. Calculating $[[L, \bar{K}]](\partial_i, \partial_j)$ for $i, j > 1$ we get the system

$$\frac{\partial \bar{K}^{\alpha+1}_j}{\partial x^i} - \frac{\partial \bar{K}^{\alpha+1}_i}{\partial x^j} - \frac{\partial \bar{K}^{\alpha}_j}{\partial x^{i-1}} + \frac{\partial \bar{K}^{\alpha}_i}{\partial x^{j-1}} = 0, \quad \alpha = 1, \dots, n-2,$$
$$\frac{\partial \bar{K}^{n-1}_j}{\partial x^{i-1}} - \frac{\partial \bar{K}^{n-1}_i}{\partial x^{j-1}} = 0.$$

Repeating process, we get, that for $[[K, L]] = 0$ is decomposed as $K = M + \mathcal{L}_\xi L$ for some ξ where

$$M = \begin{pmatrix} a^1 & 0 & \dots & 0 \\ a^2 & 0 & \dots & 0 \\ & & \dots & \\ a^n & 0 & \dots & 0 \end{pmatrix}$$

Nijenhuis cohomology in case of Jordan block

For M we have that for $i, j > 1$

$$\begin{aligned} & [[L, M]](\partial_{x^i}, \partial_{x^j}) = \\ = & L[M\partial_{x^i}, \partial_{x^j}] + L[\partial_{x^i}, M\partial_{x^j}] - [M\partial_{x^i}, L\partial_{x^j}] - [L\partial_{x^i}, M\partial_{x^j}] = \\ = & 0. \end{aligned}$$

Thus, the vanishing of the bracket is equivalent to the following for $j > 1$

$$\begin{aligned} & [[L, M]](\partial_{x^1}, \partial_{x^j}) = L[M\partial_{x^1}, \partial_{x^j}] - [M\partial_{x^1}, L\partial_{x^j}] = \\ = & L[a^\alpha \partial_{x^\alpha}, \partial_{x^j}] - [a^\alpha \partial_{x^\alpha}, \partial_{x^{j-1}}] = 0. \end{aligned}$$

This yields the system

$$\begin{aligned} & \frac{\partial a^{\alpha+1}}{\partial x^j} - \frac{\partial a^\alpha}{\partial x^{j-1}} = 0, \quad \alpha = 1, \dots, n-1, \\ & \frac{\partial a^n}{\partial x^{j-1}} = 0. \end{aligned}$$

The last equation is exactly the condition, that for $\eta = M\partial_{x^1}$ one has $\mathcal{L}_\eta L = 0$. Thus, $\mathcal{H}^1_{FN}$ is parametrized by n functions of single variable.

The Frolicher-Nijenhuis torsion

Consider a pair of quasi-linear PDEs

$$u_t^\alpha = A_q^\alpha u_x^q, \quad u_\tau^\alpha = B_q^\alpha u_x^q.$$

The solution to this system is a vector function $u(t,\tau,x)$. The system is obviously over determined, thus, the compatibility conditions appear.

$$\begin{aligned}
\partial_\tau\left(A_\beta^c u_n^\beta\right) - \partial_t\left(B_\beta^c u_n^\beta\right) &= \partial_\alpha A_\beta^c u_\tau^\alpha u_n^\beta + A_\beta^c u_{\tau n}^\beta - \partial_\alpha B_\beta^c u_t^\alpha u_n^\beta - B_\beta^c u_{tn}^\beta = \\
&= \partial_\alpha A_\beta^c B_\gamma^\alpha u_n^\gamma u_n^\beta + A_\beta^c \partial_n\left(B_\gamma^\beta u_n^\gamma\right) - \partial_\alpha B_\beta^c A_\gamma^\alpha u_n^\gamma u_n^\beta - B_\beta^c \partial_n\left(A_\gamma^\beta u_n^\gamma\right) = \\
&= \underbrace{\left(\partial_\alpha A_\beta^c B_\gamma^\alpha + A_\beta^c \partial_\alpha B_\gamma^\beta - \partial_\alpha B_\beta^c A_\gamma^\alpha - B_\beta^c \partial_\alpha A_\gamma^\beta\right) u_n^\alpha u_n^\gamma}_{1} + \\
&+ \underbrace{\left(A_\beta^c B_\gamma^\beta - B_\beta^c A_\gamma^\beta\right) u_{nn}^\gamma}_{2}.
\end{aligned} \tag{2}$$

We get thatthe compatibility condition requires that A, B commute as operator fields and the

$$\partial_\alpha A_\beta^c B_\gamma^\alpha + A_\beta^c \partial_\alpha B_\gamma^\beta - \partial_\alpha B_\beta^c A_\gamma^\alpha - B_\beta^c \partial_\alpha A_\gamma^\beta = 0.$$

The Frolicher-Nijenhuis torsion

One can show that the l.h.s. of the last equation defines a symmetric tensor. But we take different approach. The following formula appeared in the original calculations of Nijenhuis

$$\mathcal{N}_{L,K}(\xi,\eta) = L[\xi, K\eta] + K[L\xi,\eta] - [L\xi, K\eta] - KL[\xi,\eta] \tag{3}$$

This formula reminds the half of the Frolicher-Nijenhuis bracket of L, K. But does it define a tensor field? To check that we need to check the linearilty over C^∞. This yields:

$$\begin{aligned} &\mathcal{N}_{L,K}(\xi, f\eta) = \\ =&L[\xi, fK\eta] + K[L\xi, f\eta] - [L\xi, fK\eta] - KL[\xi, f\eta] = \\ =&f\mathcal{N}_{L,K}(\xi, f\eta) + \mathcal{L}_\xi fLK\eta + \mathcal{L}_{L\xi} fK\eta - \mathcal{L}_{L\xi} fK\xi - \mathcal{L}_\xi fKL\eta = \\ =&\mathcal{L}_\xi f(LK - KL)\xi \end{aligned}$$

Thus, the formula (3) defines a tensor field if and only if $LK - KL = 0$. The corresponding torsion for pair of commuting operators we call **Frolicher-Njienhuis torsion**.

The Frolicher-Nijenhuis torsion

The Frolicher-Nijenhuis torsion is neither symmetric nor skew-symmetric in lower indices. As it has two indices, it is uniquely decomposed into the sum of symmetric and skew-symetric part.

The skew-symmerization yields

$$
\begin{aligned}
&L[\xi, K\eta] + K[L\xi, \eta] - [L\xi, K\eta] - KL[\xi, \eta] - \\
-&L[\eta, K\xi] - K[L\eta, \xi] + [L\eta, K\xi] + KL[\eta, \xi] = \\
=&[[L, K]](\xi, \eta)
\end{aligned}
$$

Thus, the skew-symmetric part is just Frolicher-Nijenhuis bracket of L and K. The symmetric part is

$$
\begin{aligned}
&L[\xi, K\eta] + K[L\xi, \eta] - [L\xi, K\eta] - KL[\xi, \eta] + \\
+&L[\eta, K\xi] + K[L\eta, \xi] - [L\eta, K\xi] - KL[\eta, \xi] = \\
=&T(\xi, \eta)
\end{aligned}
$$

The Frolicher-Nijenhuis torsion

Vanishing of $T = 0$ is equivalent to the quadratic condition: for arbitrary ξ one has

$$L[\xi, K\xi] + K[L\xi, \xi] - [L\xi, K\xi] = 0.$$

Direct computation yields exactly the condition of compatibility.