Nijenhuis Geometry

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Lecture 18:
Nijenhuis pencils and
second cohomology theory

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Spring 2021

Nijenhuis pencils

The subspace in $\mathcal{P} \subset \Psi^1$ is called **Nijenhuis pencil** if the restriction of Frolicher-Nijenhuis bracket on $\mathcal{P}$ is zero. The space can be finite or infinite dimensional.

Example: Fix local coordinates and consider infinite dimensional subspace, consisting of operator fields

$$L = \begin{pmatrix} h_1(x^1) & 0 & \dots & 0 \\ 0 & h_2(x^2) & \dots & 0 \\ & & \ddots & \\ 0 & 0 & \dots & h_n(x^n) \end{pmatrix}$$

This is an infinite dimensional Nijenhuis pencil. The parameters here are n functions of single variable.

Nijenhuis pencils

Example: Consider Nijenhuis operator L and analytic functions f. We known that for arbitrary f, g we have

$$[[f(L), g(L)]] = 0.$$

This is the infinite dimensional Nijenhuis pencil, parametrized by a single analytic function of one variable.

Example: Consider a collection of constant matrices. They obviously define a Nienhuis pencil.

There is a natural class of Nijenhuis pencils, we call **maximal Nijenhuis pencils**. The pencil $\mathcal{P}$ is maximal if for any $K \in \Psi^1$ the condition $[[\mathcal{P}, K]] = 0$ implies, that $K \in \mathcal{P}$.

Nijenhuis pencils

Lemma

Consider a pair of non-degenerate Nijenhuis operators L and K. We have that $[[L, K]] = 0$ if and only if LK^{-1} is Nijenhuis operator.

Proof: We have

$$\mathcal{N}_L(\xi,\eta) + LK^{-1}LK^{-1}\mathcal{N}_K(\xi,\eta) - \mathcal{N}_{LK^{-1}}(K\xi, K\eta) =$$
$$= LK^{-1}K[L\xi,\eta] + LK^{-1}K[\xi, L\eta] - LK^{-1}KL[\xi,\eta] - [L\xi, L\eta] +$$
$$+ LK^{-1}L[K\xi,\eta] + LK^{-1}L[\xi, K\eta] - LK^{-1}LK[\xi,\eta] - LK^{-1}LK^{-1}[K\xi, K\eta] -$$
$$- LK^{-1}[L\xi, K\eta] - LK^{-1}[K\xi, L\eta] + LK^{-1}LK^{-1}[K\xi, K\eta] + [L\xi, L\eta] =$$
$$= LK^{-1}[[L, K]](\xi,\eta).$$

In case of L, K are Nijenhuis we get formula

$$-\mathcal{N}_{LK^{-1}}(K\xi, K\eta) = LK^{-1}[[L, K]](\xi,\eta).$$

The lemma is proved.

Maximal Nijenhuis pencils

Theorem

Assume that $L = \operatorname{diag}\{x^1, \dots, x^n\}$ and $[[L, K]] = [[L^2, K]] = 0$. Then K is diagonal with each diagonal component depending on its coordinate only.

Proof: From the previous calculations we know, that $[[L, K]] = 0$ implies, that $K^i_j = (x^i - x^j)\frac{\partial K^i_i}{\partial x^j}$, where K^i_j are arbitrary functions. Now consider $[[L^2, K]] = 0$. We get

$$
\begin{aligned}
[[L^2, K]](\partial_{x^i}, \partial_{x^j}) &= L^2[K\partial_{x^i}, \partial_{x^j}] + L^2[\partial_{x^i}, K\partial_{x^j}] - [L^2\partial_{x^i}, K\partial_{x^j}] - [K\partial_{x^i}, L^2\partial_{x^j}] = \\
&= -(x^\alpha)^2 \frac{\partial K^\alpha_i}{\partial x^j}\partial_{x^\alpha} - (x^\alpha)^2 \frac{\partial K^\alpha_j}{\partial x^i}\partial_{x^\alpha} + (x^i)^2 \frac{\partial K^\alpha_j}{\partial x^i}\partial_{x^\alpha} + \\
&+ 2x^i K^i_j \partial_{x^i} + (x^j)^2 \frac{\partial K^\alpha_i}{\partial x^j}\partial_{x^\alpha} - 2x^j K^j_i \partial_{x^j} = \\
&= \left((x^j - x^i)(x^j + x^i)\frac{\partial K^i_i}{\partial x^j} + 2x^i K^i_j\right)\partial_{x^i} + \dots .
\end{aligned}
\tag{1}
$$

Maximal Nijenhuis pencils

Substituting the first condition into the second, we get

$$(x^j - x^i)^2 \frac{\partial K^i_i}{\partial x^j} = 0.$$

To complete the proof one needs to check that any two such operators commute with respect to Frolicher-Nijenhuis bracket. This follows from the identity

$$2[[L, K]] = \mathcal{N}_{L+K} - \mathcal{N}_L - \mathcal{N}_K$$

Thus, we get Nijenhuis pencil and it is maximal. ■

Theorem

Consider a linear space $\mathcal{P}$ of operators L of the form

$$L = A + xc^T,$$

where A is constant matrix, columns $x = (x^1, \dots, x)$, $c = (c^1, \dots c^n)$. The space has dimension $n^2 + n$. The space is Nijenhuis pencil and it is a maximal pencil.

Proof of theorem

Consider operator field $B^i_j = x^i c^j$ and constant operator field A^i_j. Then we have For $i \neq j$ consider

$$[[A, B]](\partial_{x^i}, \partial_{x^j}) =$$
$$= A[B\partial_{x^i}, \partial_{x^j}] + A[\partial_{x^i}, B\partial_{x^j}] - [A\partial_{x^i}, B\partial_{x^j}] - [B\partial_{x^i}, A\partial_{x^j}] =$$
$$= A[c^i x^\alpha \partial_{x^\alpha}, \partial_{x^j}] + A[\partial_{x^i}, c^j x^\alpha \partial_{x^\alpha}] - [A^\alpha_i \partial_{x^\alpha}, c^j x^\alpha \partial_{x^\alpha}] - [c^i x^\alpha \partial_{x^\alpha}, A^\beta_j \partial_{x^\beta}] =$$
$$= c^i A^\alpha_j \partial_{x^\alpha} - c^j A^\alpha_i \partial_{x^\alpha} + c^j A^\alpha_i \partial_{x^\alpha} - c^i A^\alpha_j \partial_{x^\alpha} = 0.$$

Now consider

$$\frac{1}{2}[[B, B]](\partial_{x^i}, \partial_{x^j}) = B[B\partial_{x^i}, \partial_{x^j}] + B[\partial_{x^i}, B\partial_{x^j}] - [B\partial_{x^i}, D\partial_{x^j}] =$$
$$= B[c^i x^\alpha \partial_{x^\alpha}, \partial_{x^j}] + B[\partial_{x^i}, c^j x^\alpha \partial_{x^\alpha}] - [c^i x^\alpha \partial_{x^\alpha}, c^j x^\alpha \partial_{x^\alpha}] =$$
$$= -c^i c^j x^\alpha \partial_{x^\alpha} + c^i c^j x^\alpha \partial_{x^\alpha} = 0.$$

Thus, B itself is a Nijenhuis operator.

Proof of theorem

The Nijenhuis condition for $A + B$ takes the form

$$[[A + B, A + B]] = 2[[A, B]] + [[B, B]] = 0.$$

Thus, the space we have described is indeed Nijenhuis pencil.

Now consider the linear space $\bar{\mathcal{P}}$ of constant operators. It is obviously Nijenhuis pencil. Consider the centralizer $C(\bar{\mathcal{P}})$, which consists of all operator fields K, such that $[[\bar{\mathcal{P}}, K]] = 0$. We have that $\bar{\mathcal{P}} \subseteq C(\bar{\mathcal{P}})$.

Fix i, j and consider $A = \partial_{x^i} \otimes \mathrm{d}\, x^j$. By definition $A\partial_{x^k} = \partial_{x^i}\delta^k_j$. All such matrices form a basis in $\bar{\mathcal{P}}$. For arbitrary operator field we have

$$\begin{aligned} &[[A, K]](\partial_{x^k}, \partial_{x^s}) = \\ =&A[K^\alpha_k \partial_{x^\alpha}, \partial_{x^s}] + A[\partial_{x^k}, K^\alpha_s \partial_{x^\alpha}] - \delta^k_j[\partial_{x^i}, K^\alpha_s \partial_{x^\alpha}] - \delta^s_j[K^\alpha_k \partial_{x^\alpha}, \partial_{x^i}] = \\ =&\frac{\partial K^j_s}{\partial x^k}\partial_{x^i} - \frac{\partial K^j_k}{\partial x^s}\partial_{x^i} - \delta^k_j \frac{\partial K^\alpha_s}{\partial x^i}\partial_{x^\alpha} + \delta^s_j \frac{\partial K^\alpha_k}{\partial x^i}\partial_{x^\alpha} = 0. \end{aligned}$$

Proof of Theorem

Consider $j = k$. We get

$$0 = \Big(\frac{\partial K_s^j}{\partial x^j} - \frac{\partial K_j^j}{\partial x^s} - \frac{\partial K_s^i}{\partial x^i}\Big)\partial_{x^i} + \sum_{\alpha \neq i} \frac{\partial K_s^\alpha}{\partial x^i}\partial_{x^\alpha} = 0.$$

The last equation implies, that $m-$th row of K must depend only on x^m. The first equation implies, that

$$\frac{\partial K_s^j}{\partial x^j} - \frac{\partial K_s^i}{\partial x^i} = 0.$$

As K_s^j is a function of x^j and K_s^i is a function of x^i we get that identity holds if and only if the derivatives are constants. In particular the derivatives in columns coincide. Naming them c^j we get, that

$$K_j^i = A_j^i + x^i c^j.$$

We have shown, that $C(\bar{\mathcal{P}}) = \mathcal{P}$. If K is such that $[[K, \mathcal{P}]] = 0$, then $[[K, \bar{\mathcal{P}}]] = 0$ and $K \in C(\bar{\mathcal{P}}) = \mathcal{P}$. That is the $\mathcal{P}$ is maximal pencil. ■

Nijenhuis pencil

We get the following:

1. Maximal pencils can be both finite and infinite dimensional
2. The constant operators do not form a maximal pencil (quite surprizing)

Theorem

For $n > 2$ consider the space $\mathcal{P}$ of the operators L in the form

$$L = A + xb^T + bx^T + Kxx^T,$$

where constant matrix $A^T = A$, columns $x = (x^1, \dots, x^n)$, $b = (b^1, \dots, b^n)$ and K is arbitrary constant. Then the corresponding space is maximal Nijenhuis pencil.

Proof of theorem

Consider $\mathcal{S}$ to be the Nijenhuis pencil of constant symmetric matrices and consider $C(\mathcal{S})$. Constant matrices contain all diagonal matrices with pairwise distinct $\lambda_1, \dots, \lambda_n$ on the diagonal.

From description of $\mathcal{H}^1_{FN}$ we know that for diagonal operator L the condition $[[K, L]] = 0$ yields: K^i_i depends on x^i and

$$(\lambda_\alpha - \lambda_i)\frac{\partial K^\alpha_j}{\partial x^i} - (\lambda_\alpha - \lambda_j)\frac{\partial K^\alpha_i}{\partial x^j} = 0$$

for $\alpha \neq i, \neq j$. This condition holds for all α_i, α_j, which implies that $\frac{\partial K^\alpha_i}{\partial x^j} = 0$ for triple of pairwise different i, j, α.

This implies, that K^α_i depends only on x^α, x^i

Proof of theorem

Now consider $A = \partial_{x^i} \otimes \mathrm{d}\, x^j + \partial_{x^j} \otimes \mathrm{d}\, x^i$. The formulas for $[[A, K]] = 0$ follow from earlier calculations and yield

$$
\begin{aligned}
&[[A, K]](\partial_{x^k}, \partial_{x^s}) = \\
=&\frac{\partial K_s^j}{\partial x^k}\partial_{x^i} - \frac{\partial K_k^j}{\partial x^s}\partial_{x^i} - \delta_j^k \frac{\partial K_s^\alpha}{\partial x^i}\partial_{x^\alpha} + \delta_j^s \frac{\partial K_k^\alpha}{\partial x^i}\partial_{x^\alpha} + \\
+&\frac{\partial K_s^i}{\partial x^k}\partial_{x^j} - \frac{\partial K_k^i}{\partial x^s}\partial_{x^j} - \delta_i^k \frac{\partial K_s^\alpha}{\partial x^j}\partial_{x^\alpha} + \delta_i^s \frac{\partial K_k^\alpha}{\partial x^j}\partial_{x^\alpha}
\end{aligned}
$$

For i, j, k, s pairwise different this formula yields zero. Assume that $k = j$ and $s \neq i$. The formula yields

$$
\frac{\partial K_s^j}{\partial x^j} - \frac{\partial K_s^i}{\partial x^i} = 0.
$$

We get that $K_s^j = f^s(x^s)x^j$ (in previous calculations we had f^s being constants).

Proof of theorem

Finally consider $k = j, s = i$. We get

$$0 = \sum_{\alpha \neq i,j} \Big(\frac{\partial K_j^\alpha}{\partial x^j} - \frac{\partial K_j^\alpha}{\partial x^j} \Big) \partial_{x^\alpha} = 0.$$

This implies, that $(f^j)' = (f^i)'$. Integrating, we get, that $[[A, K]] = 0$ implies, that

$$K_j^i = A_j^i + c^i x^j + x^i b^j + K x^i x^j.$$

Now consider space $\bar{\mathcal{S}}$ which is spanned by constant symmetric operators $A = A^T$ and $B_j^i = x^i x^j$. Consider

$$\begin{aligned} &[[B, B]](\partial_{x^i}, \partial_{x^j}) = \\ =&B[B\partial_{x^i}, \partial_{x^j}] + B[\partial_{x^i}, B\partial_{x^j}] - [B\partial_{x^i}, B\partial_{x^j}] = \\ =&B[x^i x^\alpha \partial_{x^\alpha}, \partial x^j] + B[\partial_{x^i}, x^j x^\alpha \partial_{x^\alpha}] - [x^i x^\alpha \partial_{x^\alpha}, x^j x^\alpha \partial_{x^\alpha}] = \\ =&x^i x^j x^\alpha \partial_{x^\alpha} - x^j x^i x^\alpha \partial_{x^\alpha} - x^i x^j x^\alpha \partial_{x^\alpha} + x^j x^i x^\alpha \partial_{x^\alpha} = 0. \end{aligned}$$

Proof of theorem

Thus, we get that $\bar{\mathcal{S}}$ is Nijenhuis pencil itself. A the same time for $K \neq 0$ we can write

$$L = A + xb^T + bx^T + Kxx^T = A - \frac{1}{K^2}bb^T + K(x + \frac{1}{K}b)(x + \frac{1}{K}b)^T.$$

We get that L is Nijenhuis for $K \neq 0$ and, thus, it is Nijenhuis for all K by continuity. Consider $C^i_j = p^i x^j$ we get

$$[[C, B]](\partial_{x^i}, \partial_{x^j}) = C[B\partial_{x^i}, \partial_{x^j}] + C[\partial_{x^i}, B\partial_{x^j}] + B[C\partial_{x^i}, \partial_{x^j}] + B[\partial_{x^i}, C\partial_{x^j}] -$$
$$- [C\partial_{x^i}, B\partial_{x^j}] - [B\partial_{x^i}, C\partial_{x^j}] =$$
$$= C[x^i x^\alpha \partial_{x^\alpha}, \partial_{x^j}] + C[\partial_{x^i}, x^j x^\alpha \partial_{x^\alpha}] + B[x^i p^\alpha \partial_{x^\alpha}, \partial_{x^j}] + B[\partial_{x^i}, x^j p^\alpha \partial_{x^\alpha}] -$$
$$- [x^i p^\alpha \partial_{x^\alpha}, x^j x^\beta \partial_{x^\beta}] - [x^i x^\alpha \partial_{x^\alpha}, x^j p^\beta \partial_{x^\beta}] =$$
$$= C(x^i \partial_{x^j} - x^j \partial_{x^i}) - \delta^i_\beta x^j x^\beta p^\alpha \partial_{x^\alpha} + x^i p^\alpha \delta^j_\alpha x^\beta \partial_{x^\beta} + x^i p^\alpha x^j \delta^\beta_\alpha \partial_{x^\beta} -$$
$$- x^j p^\beta \delta^i_\beta x^\alpha \partial_{x^\alpha} - x^i x^j p^\beta \delta^\alpha_\beta \partial_{x^\alpha} + x^i x^\alpha \delta^j_\alpha p^\beta \partial_{x^\beta} =$$
$$= -x^j x^i p^\alpha \partial_{x^\alpha} + x^i p^j x^\alpha \partial_{x^\alpha} + x^i x^j p^\alpha \partial_{x^\alpha} - x^j p^i x^\alpha \partial_{x^\alpha} -$$
$$- x^i x^j p^\alpha \partial_{x^\alpha} + x^i x^j p^\alpha \partial_{x^\alpha} = (x^i p^j - x^j p^i) x^\alpha \partial_{x^\alpha}$$

Proof of theorem

Now consider $C(\bar{\mathcal{S}})$. As $\mathcal{S} \subset \bar{\mathcal{S}}$, we get that elements $K \in C(\bar{\mathcal{S}})$ are in the form

$$K = A + xc^T + bx^T + Kxx^T.$$

Due to the graded nature of Frolicher-Nijenhuis bracket we get, that $[[K, B]] = 0$ implies, that

$$K = A + xb^T + bx^T + Kxx^T.$$

We have shown, that all such operators are Nijenhuis and $\bar{\mathcal{S}} \in \mathcal{P}$. Thus, we get maximal Nijenhuis pencil.

Another cohomology

Recall, that we have defined $\mathrm{d}_L : \Psi^k \to \Psi^{k+1}$ as $\mathrm{d}_L K = [[K, L]]$. Now consider

$$\mathcal{L}_L = i_L \mathrm{d} - \mathrm{d}\, i_L.$$

This operation acts as

$$\mathcal{L}_L : \Omega^k \to \Omega^{k+1}.$$

Lemma

For Nijenhuis operator L we have $\mathcal{L}_L^2 = 0$

Proof: First, let us prove that for $f \in \Omega^0$. As $\mathcal{L}_L(f) = L^* \mathrm{d} f$ we get

$$\mathcal{L}_L^2 f = i_L \mathrm{d}\,(L^* \mathrm{d} f) - \mathrm{d}\, i_L (L^* \mathrm{d} f)$$

Substituting vector fields ξ, η we get

$$\mathcal{L}_L^2 f(\xi, \eta) = L^* \mathrm{d} f(L\xi, \eta) + L^* \mathrm{d} f(\xi, L\eta) - \mathrm{d}\,((L^*)^2 \mathrm{d} f)(\xi, \eta) = 0.$$

This is exactly the vanishing of Nijenhuis torsion on $1-$forms

Another cohomology

Now, recall, that $[\mathrm{d}\,, \mathcal{L}_L] = 0$ for graded Lie-bracket. In particular, this implies, that

$$\mathrm{d}\,\mathcal{L}_L = \mathcal{L}_L \mathrm{d}\,.$$

Thus, we get that $\mathcal{L}_L \mathrm{d}\, f = \mathrm{d}\,(\mathcal{L}_L f) = 0$. This implies, that $\mathcal{L}_L^2 = 0$ on exact $1-$form. We know, that in local coordinates Ω is generated by Ω^0 and exact $1-$forms. Thus, as $\mathcal{L}_L$ is graded derivation, we get that the Lemma holds. ■

Note, that in case $L = \mathsf{Id}$ we get that $\mathcal{L}_L = \mathrm{d}\,$. Indeed, it acts as $\mathrm{d}\,$ on Ω^0 and $\mathcal{L}_{\mathsf{Id}} \mathrm{d}\, f = 0$ by definition. Thus, this operation coincides with $\mathrm{d}\,$.

In terms of this cohomology, we call **Nijenhuis cohomology**, the conservation laws of L are exactly $f \in \Omega^0$ with property

$$\mathrm{d}\,\mathcal{L}_L f = 0.$$

Exercises

1. Proof the the maximality of pencil

$$L = A + bx^T + xb^T + Kxx^T$$

   in dimension two.
2. Prove that $\mathcal{H}_N^0$ for non-degenerate L conincides with de Rham cohomology
3. Calculate Nijenhuis cohomology $\mathcal{H}_N^1$ for diagonal Nijenhuis operator with constant pairwise distinct eigenvalues