Nijenhuis Geometry

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Lecture 19:
One cohomology lemma
and some applications to PDEs

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Recollecting facts

We have proved that $\mathcal{L}_L : \Omega^k \to \Omega^{k+1}$ defines a differentiation with the following properties (they uniquely define the operation):

1. $\mathcal{L}_L$ is linear operation
2. $\mathcal{L}_L f = L^* \mathrm{d}\, f$
3. $\mathcal{L}_L \mathrm{d} + \mathrm{d}\, \mathcal{L}_L = 0$
4. $\mathcal{L}_L(\alpha \wedge \beta) = \mathcal{L}_L \alpha \wedge \beta + (-1)^k \alpha \wedge \mathcal{L}_L \beta$ for $\alpha \in \Omega^k, \beta \in \Omega^m$

The last condition implies, that

$$
\begin{aligned}
&\mathcal{L}_L^2(\alpha \wedge \beta) = \mathcal{L}_L\big(\mathcal{L}_L \alpha \wedge \beta + (-1)^k \alpha \wedge \mathcal{L}_L \beta\big) = \\
=&\mathcal{L}_L^2 \alpha \wedge \beta + (-1)^{k+1} \mathcal{L}_L \alpha \wedge \mathcal{L}_L \beta + (-1)^k \mathcal{L}_L \alpha \wedge \mathcal{L}_L \beta + \alpha \wedge \mathcal{L}_L^2 \beta
\end{aligned}
$$

As Ω is generated by exact forms and Ω^0, then together with third condition we get: $\mathcal{L}_L^2 = 0$ if and only if $\mathcal{L}_L^2 : \Omega^0 \to 0$. The latter is equivalent to the vanishing of Nijenhuis torsion.

Recollecting facts

Consider operation

$$D = \mathcal{L}_L \mathrm{d} \, : \Omega^k \to \Omega^{k+2}$$

By construction, D is not a derivation of algebra Ω as it does not respect the wedge product.

At the same time if we treat Ω^{2k} as a complex, then we have cohomology defined, using this operation. We get that the closed elements of Ω^0 in this case are functions

$$\mathcal{L}_L \mathrm{d}\, f = (\mathrm{d}\, i_L - i_L \mathrm{d}\,)\mathrm{d}\, f = \mathrm{d}\,(i_L \mathrm{d}\, f) = \mathrm{d}\,(L^* \mathrm{d}\, f) = 0.$$

Thus, the closed $0-$forms are exactly the conservation laws for L.

Conservation laws in diagonal case

Lemma

For $L = \mathrm{diag}(x^1, \ldots, x^n)$ *the conservation laws* f *have the form*

$$f = f^1(x^1) + \cdots + f^n(x^n)$$

Proof: We have that

$$L^* \mathrm{d}\, f = x^1 \frac{\partial f^1}{\partial x^1} \mathrm{d}\, x^1 + \cdots + x^n \frac{\partial f^n}{\partial x^n} \mathrm{d}\, x^n.$$

Differentiating we get

$$\mathrm{d}\left(L^* \mathrm{d}\, f\right) = \sum_{i<j} \left(x^j - x^i\right) \frac{\partial^2 f}{\partial x^i \partial x^j} \mathrm{d}\, x^i \wedge \mathrm{d}\, x^j$$

This implies, that all the derivatives

$$\frac{\partial^2 f}{\partial x^i \partial x^j} = 0.$$

The Lemma is proved.

The cohomological equation

Theorem

For differentially non-degenerate operator L consider equation

$$\mathcal{L}_L \mathrm{d} f = \omega.$$

It locally has solution if and only if $\mathrm{d}\,\omega = \mathcal{L}_L \omega = 0$.

Proof: The necessity of the condition follows from the properties of the differentiations. Now, assume that L is diagonal with x^i on the diagonal and that both conditions hold. First, we have

$$\mathcal{L}_L \omega = \left(\mathrm{d}\, i_L - i_L \mathrm{d}\,\right)\omega = \mathrm{d}\,(i_L \omega) = 0.$$

In other words, we have that $i_L \omega$ is closed. Recall, that

$$i_L \omega(\xi, \eta) = \omega(L\xi, \eta) + \omega(\xi, L\eta).$$

Denote

$$\omega = \sum_{i<j} \omega_{ij}(x) \mathrm{d}\, x^i \wedge \mathrm{d}\, x^j$$

Proof of theorem

At the same time

$$i_L\omega = \sum_{i<j}(x^i + x^j)\omega_{ij}(x)\mathrm{d}\,x^i \wedge \mathrm{d}\,x^j.$$

The equation $\mathcal{L}_L \mathrm{d}\,f = \omega$ is written as

$$(x^i - x^j)\frac{\partial^2 f}{\partial x^i \partial x^j} = \omega_{ij}$$

It can be rewritten as

$$\frac{\partial^2 f}{\partial x^i \partial x^j} = \frac{\omega_{ij}}{x^i - x^j}.$$

This is a regular cohomology PDE. It has a local solution if the r.h.s. defines a closed 1−form.

Proof of theorem

In local coordinates it can be written as

$$\frac{\partial}{\partial x^k}\left(\frac{\omega_{ij}}{x^i - x^j}\right) = \frac{\partial}{\partial x^i}\left(\frac{\omega_{jk}}{x^j - x^k}\right) = \frac{\partial}{\partial x^j}\left(\frac{\omega_{ki}}{x^k - x^i}\right)$$

for $i \neq j \neq k \neq i$, which can also be rewritten as

$$\frac{\partial \omega_{ij}}{\partial x^k}\frac{1}{(x^i - x^j)} = \frac{\partial \omega_{jk}}{\partial x^i}\frac{1}{(x^j - x^k)} = \frac{\partial \omega_{ki}}{\partial x^j}\frac{1}{(x^k - x^i)}$$

In the same terms the conditions $\mathrm{d}\,\omega = 0$ and $\mathrm{d}\, i_L\omega = 0$ take form

$$\begin{aligned} &\frac{\partial \omega_{ij}}{\partial x^k} + \frac{\partial \omega_{jk}}{\partial x^i} + \frac{\partial \omega_{ki}}{\partial x^j} = 0, \\ &\frac{\partial \omega_{ij}}{\partial x^k}(x^i + x^j) + \frac{\partial \omega_{jk}}{\partial x^i}(x^j + x^k) + \frac{\partial \omega_{ki}}{\partial x^j}(x^k + x^i) = 0 \end{aligned}$$

Dividing the second equation by $\frac{1}{x^i + x^j}$ and subtracting from the first one, we get

Proof of theorem

$$
\begin{aligned}
0 &= \frac{\partial \omega_{jk}}{\partial x^i}\left(1 - \frac{x^j + x^k}{x^i + x^j}\right) + \frac{\partial \omega_{ki}}{\partial x^j}\left(1 - \frac{x^k + x^i}{x^i + x^j}\right) = \\
&= \frac{\partial \omega_{jk}}{\partial x^i}\frac{x^i - x^k}{x^i + x^j} - \frac{\partial \omega_{ki}}{\partial x^j}\frac{x^k - x^j}{x^i + x^j}
\end{aligned}
$$

This is exactly the condition we were looking for. Thus, $\mathrm{d}\,\omega = 0$ and $\mathcal{L}_L\omega = 0$ imply that there exist function f such, that $\mathcal{L}_L \mathrm{d}\, f = \omega$ for diagonal L.

Now assume that in coordinates $x^1, \dots, x^n$ we have

$$
L = \begin{pmatrix}
x^1 & 1 & 0 & \dots & 0 \\
x^2 & 0 & 1 & \dots & 0 \\
 & & & \ddots & \\
x^{n-1} & 0 & 0 & \dots & 1 \\
x^n & 0 & 0 & \dots & 0
\end{pmatrix}
$$

Proof of theorem

We have

$$\mathcal{L}_L \mathrm{d} f = \mathrm{d}(i_L \mathrm{d} f) = \mathrm{d}\left(\Big(\sum_\alpha \frac{\partial f}{\partial x^\alpha} x^\alpha\Big)\mathrm{d} x^1 + \frac{\partial f}{\partial x^1}\mathrm{d} x^2 + \cdots + \frac{\partial f}{\partial x^{n-1}}\mathrm{d} x^n\right) =$$

$$= \sum_{j>1}\left(\frac{\partial^2 f}{\partial x^1 \partial x^{j-1}} - \frac{\partial f}{\partial x^j} - \sum_\alpha \frac{\partial^2 f}{\partial x^\alpha \partial x^j} x^\alpha\right)\mathrm{d} x^1 \wedge \mathrm{d} x^j +$$

$$+ \sum_{1<i<j}\left(\frac{\partial^2 f}{\partial x^i \partial x^{j-1}} - \frac{\partial^2 f}{\partial x^{i-1}\partial x^j}\right)\mathrm{d} x^i \wedge \mathrm{d} x^j.$$

This yields the system of PDEs

$$\frac{\partial^2 f}{\partial x^1 \partial x^{j-1}} - \frac{\partial f}{\partial x^j} - \sum_\alpha \frac{\partial^2 f}{\partial x^\alpha \partial x^j} x^\alpha = \omega_{1j}, \quad j = 2, \ldots, n$$

$$\frac{\partial^2 f}{\partial x^i \partial x^{j-1}} - \frac{\partial^2 f}{\partial x^{i-1}\partial x^j} = \omega_{ij}, \quad 2 \le i < j \le n.$$

This system looks horrifying!

Proof of theorem

Consider the first equation. For $j = n$ we get

$$\frac{\partial^2 f}{\partial x^1 \partial x^{n-1}} = F_{1n-1}\left(x, \frac{\partial f}{\partial x^j}, \frac{\partial^2 f}{\partial x^i \partial x^n}\right)$$

For $j = n - 1$ we get

$$\frac{\partial^2 f}{\partial x^1 \partial x^{n-2}} = \omega_{1n-1} + \frac{\partial f}{\partial x^{n-1}} + \frac{\partial^2 f}{\partial x^1 \partial x^{n-1}} x^1 + \sum_{1<\alpha} \frac{\partial^2 f}{\partial x^\alpha \partial x^{n-1}} x^\alpha =$$

$$= \omega_{1n-1} + \frac{\partial f}{\partial x^{n-1}} + \frac{\partial^2 f}{\partial x^1 \partial x^{n-1}} x^1 + \sum_{1<\alpha} \left(\frac{\partial^2 f}{\partial x^{\alpha-1} \partial x^n} x^\alpha + \omega_{\alpha n}\right)$$

In the last step we have used the second equation for $i = \alpha, j = n$, that is

$$\frac{\partial^2 f}{\partial x^\alpha \partial x^{n-1}} - \frac{\partial^2 f}{\partial x^{\alpha-1} \partial x^n} = \omega_{in}$$

Proof of theorem

Thus, we get that

$$\frac{\partial^2 f}{\partial x^1 \partial x^{n-2}} = F_{1n-2}\Big(x, \frac{\partial f}{\partial x^j}, \frac{\partial^2 f}{\partial x^i \partial x^n}\Big)$$

Continuing this procedure, we get that

$$\frac{\partial^2 f}{\partial x^1 \partial x^j} = F_{1j}\Big(x, \frac{\partial f}{\partial x^j}, \frac{\partial^2 f}{\partial x^i \partial x^n}\Big), \quad j = 1, \dots, n-1.$$

Using the second collection of the equations for $i > 1, j > 1$, we get

$$\frac{\partial^2 f}{\partial x^i \partial x^{j-1}} = \frac{\partial^2 f}{\partial x^{i-1} \partial x^j} + \omega_{ij} = \frac{\partial^2 f}{\partial x^{i-2} \partial x^{j+1}} + \omega_{i-1\,j+1} + \omega_{ij} =$$
$$= \frac{\partial^2 f}{\partial x^1 \partial x^{i+j-2}} + \omega_{1\,j+i-1} + \omega_{2\,i+j-2} + \dots + \omega_{ij}$$

Proof of theorem

This analysis shows that (at least some) of the equations yield

$$\frac{\partial^2 f}{\partial x^i \partial x^j} = F_{ij}\Big(x, \frac{\partial f}{\partial x^j}, \frac{\partial^2 f}{\partial x^i \partial x^n}\Big), \quad 1 \le i \le j \le n-1.$$

Recall that studying the companion form we have encountered similar system of PDEs. The initial conditions here are

$$\begin{aligned} & f(0,\dots,0,x^n) = v^1(x^n), \\ & \frac{\partial f}{\partial x^j}(0,\dots,0,x^n) = v^{j+1}(x^n), \quad j = 1,\dots,n-1. \end{aligned}$$

In analytic category the system has a solution for every initial condition if and only if the compatibility conditions hold

$$\partial_k F_{ij} = \partial_i F_{ki} = \partial_j F_{jk}$$

Note, that each function F_{ij} can be written as

$$F_{ij} = A^s_{ij} \frac{\partial^2 f}{\partial x^s \partial x^n} + B^s_{ij} \frac{\partial f}{\partial x^s} + C_{ij}.$$

Proof of theorem

The coefficients A_{ij}^s, B_{ij} depend polynomially on coordinates $x^1, \dots, x^n$. The compatibility conditions in this case can be written as

$$\partial_k F_{ij} = \frac{\partial A_{ij}^s}{\partial x^k} \frac{\partial^2 f}{\partial x^s \partial x^n} + A_{ij}^s \frac{\partial}{\partial x^n}\left(\frac{\partial^2 f}{\partial x^s \partial x^k}\right) + \frac{\partial B_{ij}^s}{\partial x^k} \frac{\partial f}{\partial x^k} + B_{ij}^s \frac{\partial^2 f}{\partial x^s \partial x^k} + \frac{\partial C_{ij}}{\partial x^k} =$$

$$= \frac{\partial A_{ij}^s}{\partial x^k} \frac{\partial^2 f}{\partial x^s \partial x^n} + A_{ij}^s \frac{\partial}{\partial x^n}\left(A_{sk}^q \frac{\partial^2 f}{\partial x^q \partial x^n} + B_{sk}^q \frac{\partial f}{\partial x^q} + C_{sk}\right) +$$

$$+ \frac{\partial B_{ij}^s}{\partial x^k} \frac{\partial f}{\partial x^s} + B_{ij}^s\left(A_{sk}^q \frac{\partial^2 f}{\partial x^q \partial x^n} + B_{sk}^q \frac{\partial f}{\partial x^q} + C_{sk}\right) + \frac{\partial C_{ij}}{\partial x^k} =$$

$$= A_{ij}^s A_{sk}^q \frac{\partial^3 f}{\partial x^q \partial x^n \partial x^n} +$$

$$+ \left(\frac{\partial A_{ij}^s}{\partial x^k} + A_{ij}^s \frac{\partial A_{sk}^q}{\partial x^n} + A_{ij}^s B_{sk}^q + B_{ij}^s A_{sk}^q\right) \frac{\partial^2 f}{\partial x^q \partial x^n} +$$

$$+ \left(A_{ij}^s \frac{\partial B_{sk}^q}{\partial x^n} + \frac{\partial B_{ij}^q}{\partial x^k} + B_{ij}^s B_{sk}^q\right) \frac{\partial f}{\partial x^q} +$$

$$+ A_{ij}^s \frac{\partial C_{sk}}{\partial x^n} + B_{ij}^s C_{sk} + \frac{\partial C_{ij}}{\partial x^k}$$

Proof of theorem

We get that conditions $\partial_k F_{ij} - \partial_j F_{ki} = 0$ yield us a collection of functions, that must identically vanish: exactly the coefficients in front of partial derivatives.

We get the collection of functions

$$c_\alpha = c_\alpha\Big(A, B, C, \frac{\partial A}{\partial x}, \frac{\partial B}{\partial x}, \frac{\partial C}{\partial x}\Big).$$

These functions are anlytic. We know that in every point p for differentially non-degenerate L we have at a neighbourhood with regular points. In this neighbourhood the conditions can be rewritten in terms of diagonal operator. They identically vanish.

Thus, functions c_α are zero almost everywhere and by continuity they are zero everywhere.

Conjecture

For Nijennhuis operator L of the form

$$L = \begin{pmatrix} x^1 & 0 & 0 & \dots & 0 \\ 0 & x^2 & 0 & \dots & 0 \\ 0 & 0 & x^3 & \dots & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & \dots & x^n \end{pmatrix}$$

and arbitrary $\omega \in \Omega^k, k \geq 2$ the following conditions are equivalent:

1. $\mathrm{d}\,\omega = \mathcal{L}_L \omega = 0$
2. There exists $\alpha \in \Omega^{k-2}$ such that $\mathcal{L}_L \mathrm{d}\,\alpha = \omega$

Frolicher-Nijenhuis torsion: again

Consider a pair of systems

$$u_t = Lu_x \text{ and } u_\tau = Ku_y.$$

That is we have two spacial coordinates x, y. We get that the compatibility condition for such system can be written as

$$\begin{aligned}
0 &= \partial_t[K^\alpha_s u^s_y] - \partial_\tau[L^\alpha_s u^s_x] = \\
&= \frac{\partial K^\alpha_s}{\partial u^q} L^q_r u^s_y u^r_x + K^\alpha_s \partial_y[L^s_r u^r_x] - \frac{\partial L^\alpha_s}{\partial u^q} K^q_r u^r_y u^s_x - L^\alpha_s \partial_x[K^s_r u^r_y] = \\
&= \Big(\frac{\partial K^\alpha_r}{\partial u^q} L^q_s - \frac{\partial L^\alpha_s}{\partial u^q} K^q_r + K^\alpha_q \frac{\partial L^q_s}{\partial u^r} - L^\alpha_q \frac{\partial K^q_r}{\partial u^s}\Big) u^s_x u^r_y + \Big(K^\alpha_q L^q_s - L^\alpha_q K^q_s\Big) u^s_{xy}
\end{aligned}$$

We get that the compatibility condition yields that $\mathcal{N}_{L,K} = 0$ for commuting L, K, not only symmetric part.

Frolicher-Nijenhuis torsion: again

Lemma

Assume that L is Nijenhuis. Then $\mathcal{N}_{L^k, L^s} = 0$ for all k, s.

Proof: We have

$$\mathcal{N}_{L^k, L^s}(\xi, \eta) = L^k[\xi, L^s\eta] + L^s[L^k\xi, \eta] - L^{k+s}[\xi, \eta] - [L^k\xi, L^s\eta]$$

The vanishing of Nijenhuis torsion is

$$0 = \mathcal{L}_{L\xi} L - L\mathcal{L}_\xi L$$

Applying this identity to the Frolicher-Nijenhuis bracket, we get that it also vanishes. As a result the corresponding systems are compatible.

Frolicher-Nijenhuis torsion: again

Theorem

Consider L in dimension $2n$ to be a Nijenhuis operator, such that

1. *At the coordinate origin L is a sum of two Jordan blocks of dimension n*
2. *In the neighbourhood of the coordinate origin its characteristic polynomial is*

$$\chi_L(t) = (t^n + a_1 t^{n-1} + \cdots + a_n)^2,$$

   and $t^n + a_1 t^{n-1} + \cdots + a_n$ annihilates L.

Then there exists a coordinates, in which

$$L = \begin{pmatrix} A_{\mathrm{c}} & 0 \\ 0 & A_{\mathrm{c}} \end{pmatrix}$$

where A_{c} is a first companion matrix with first column $a_1, \ldots, a_n$.