Alexey V. Bolsinov (Loughborough, UK)
Vladimir S. Matveev (Jena, Germany)

Introduction to Nijenhuis Geometry and its applications

Lecture 3

Sydney, Feb 2022

Foreword

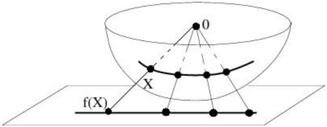
- In Lecture 1 we have seen that Nijenhuis operators appear naturally in the theory of projectively equivalent metrics
- In this lecture I show that Nijenhuis operators appear naturally in the theory of ∞-dimensional Poisson brackets of the first order.
- Morever, I show that these two theories are closely related: in particular with the help of projectively equivalent metrics of constant curvature one can construct a maximal pencil of compatible ∞-dimensional Poisson brackets of the first order, and if a pencil of compatible brackets is sufficiently big then it comes from projectively equivalent metrics. This will provide examples of infinite-dimensional integrable systems.
- The relation between projectively equivalent metrics and ∞-dimensional Poisson brackets, and also results/methods of this lecture will be explored further in Lecture 4, where we will consider compatibility of nonhomogeneous Poisson brackets of type $3+1$.

Plan of lecture 3

- Introduction: What will we use from previous lectures?
  - Projective equivalence (Lecture 1)
  - Systems of hydrodynamic type (Lecture 2)
- Hamiltonian systems of hydrodynamic type
  - Definition and formulas
  - Poisson structure of the first order
  - Jacobi identity as flatness of the metric.
  - Poisson structures of higher order
- Compatible Poisson structures
  - Compatible Poisson structures in the finite-dimensional case: Integrals and commuting vector fields
  - Infinite-dimensional case: conservative quantities and commuting flows.
  - Poisson-compatible metrics. Dubrovin condition. Relation to Nijenhuis geometry.
- Connection between projective equivalence and Poisson-compatible metrics
  - Sinjukov-Topalov hierarchy and Poisson-compatible metrics.
  - Casimirs and commuting flows
  - Two maximal pencils

What will we need from Lecture 1:

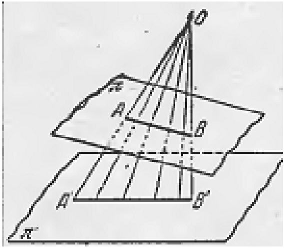
Def. Two metrics (on one manifold) are projectively equivalent if they have the same unparameterized geodesics (notation: $g \sim \bar{g}$)



Lagrange example 1789: Radial projection $f : S^2 \to \mathbb{R}^2$ takes geodesics of the sphere to geodesics of the plane, because geodesics on sphere/plane are intersection of planes containing 0 with the sphere/plane.

The example of Lagrange survives for all signatures and for all dimensions.

Instead of sphere, one can take another plane – and construct nonitrivial examples of projectively equivalent flat metrics. In fact, this construction gives us all possible examples (picture from the textbook of Yaglom).



This gives us a description of all geodesically equivalent metrics of constant curvature. Recall that by the Beltrami Theorem a metric geodesically equivalent to a metric of constant curvature is of constant curvature.

Relation of projectively equivalent metrics to Nijenhuis operators (re-cap from Lecture 1)

- Instead of the pair $(g, \bar{g})$ of metrics let us consider the pair (g, L) , where L is an operator given by

$$L^i_j := \sum_s \bar{g}^{si} g_{sj} \cdot \left| \frac{\det \bar{g}}{\det g} \right|^{\frac{1}{n+1}}.$$

  The pair (g, L) contains the same information as the pair $(g, \bar{g})$; the condition that g is projectively equivalent to $\bar{g}$ can be rewritten as a linear PDE-system on (g, L) which we call geodesic compatibility.

- **Observation (Bolsinov, Matveev 2003; discussed in the 1st lecture)**:
  - If (g, L) is geodesic compatible, then L is Nijenhuis.
  - Off-topic: This is a "mechanism" how Nijenhuis operators appear in many different unrelated subjects
  - And also hints for applications: one can start with a Nijenhuis operator L and look for the corresponding g.

Quasilinear systems of hydrodynamic type (Lecture 2)

- As mathematical objects, they are PDE systems of the form
$$\frac{\partial}{\partial t} u^i(t,x) = A(u)^i_s \frac{\partial}{\partial x} u^s(t,x). \tag{1}$$
Here $u(t,x) = (u^1(t,x), ..., u^n(t,x))$ is unknown vector-function of two variables (t,x) and A is a matrix depending on $(u^1, ..., u^n)$ with no explicit dependence on t and x.
- $(u^1, ..., u^n)$ should be viewed as local coordinate system on a manifold. The matrix A is then an operator (=$(1,1)$-tensor); after a transformation $(u^1, ..., u^n) \mapsto (\tilde{u}^1, ..., \tilde{u}^n)$ the equation (1) has the same form with $\tilde{A}(\tilde{u}) = JA(u)J^{-1}$.
- For every fixed t, $u(t,x)$ is a curve parametrised by x. By Kovalevskaya Theorem, under real-analyticity assumptions, for an initial "curve" $u(t_0,x)$ there exists, for t close to t_0, a solution $u(t,x)$ starting from $u(t_0,x)$.
- In Lecture 2 we studied the situation when A is a Nijenhuis operator L; and saw a collection of non-trivial tricks to solve it.

What are integrable ∞-dimensional Hamiltonian systems?

- In the finite-dimensional case, there is an established definition: a Hamiltonian system generated by H is (Liouville) integrable, if it has sufficiently many ($n = \frac{1}{2} \dim M$ in the most nondegenerate case) independent functions $f_1 = H, f_2, ..., f_n$ such that $\{f_i, f_j\} = 0$.
- There is no generally accepted definition in the $\infty-$dimensional case.
- Theory of integrable $\infty-$dimensional systems is a collections of methods to deal with certain very specific (systems of) PDEs, i.e., to study the behaviour and construct explicit solutions.
  - One of the most effective method is related to "bihamiltonian structures" (Magri, Morosi 1982); results of my talk are related to bi- and multiPoisson structures.
- On the next two slides I define: Hamiltonian systems of hydrodynamic type and later will speak about the integrability.

I will start with finite-dimensional systems and then mimic the steps in the ∞-dimensional case

- We work on a manifold M^n (called phase space) equipped with a Poisson structure P^{ij}:
- The Poisson bracket of two functions is an operation $\{F, H\} = \sum_{i,j} \frac{\partial F}{\partial x^i} P^{ij} \frac{\partial H}{\partial x^j}$.
- For example, in the case when $M = \mathbb{R}^{2n}(x^1, ..., x^n, p^1, ..., p^n)$ and $P^{ij} = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{-1} & \mathbf{0} \end{pmatrix}$, $\{F, H\} = \sum_i \frac{\partial F}{\partial x^i}\frac{\partial H}{\partial p^i} - \frac{\partial F}{\partial p^i}\frac{\partial H}{\partial x^i}$.
- We require that P^{ij} is skew-symmetric ($\implies$ $\{F, H\} = -\{H, F\}$ so in particular $\{H, H\} = 0$) and satisfies the Jacobi identity
- The Jacobi identity is the condition $\{\{F_1, F_2\}, F_3\} + \{\{F_2, F_3\}, F_1\} + \{\{F_3, F_1\}, F_2\} = 0$. It is a differential nonlinear condition on the components of P^{ij}.
- The Hamiltonian vector field (= generator of the Hamiltonian flow) of a function H is the vector field $X_H = \sum_s P^{sj} \frac{\partial H}{\partial x^s}$.
- The dynamical system is $\frac{d}{dt}x(t) = X_H$. It is a system of n ODEs; for every initial data $x(0)$ there locally exists a unique solution.

∞-dimensional Hamiltonian systems of hydrodynamic type

	finite dimension	infinite dimension
Background	Poisson manifold (M^n, P^{ij})	flat (M^n, g)
Phase space	Manifold M	Loop space $\{u : S^1(x) \to M\}$ "points" are closed curves $u(x)$
Hamiltonian	$H : M \to \mathbb{R}$	Functional of the form $\int_{S_1} h(u(x))dx$
Poisson structure	$\{F, H\} = \sum_{i,j} \frac{\partial F}{\partial x^i} P^{ij} \frac{\partial H}{\partial x^j}$	$\{\int_{S^1} fdx, \int_{S^1} hd\tau\} = \int_{S^1} \sum_{i,j} \frac{\delta f}{\delta u^i} \pi^{ij} \frac{\delta h}{\delta u^j} dx$, where $\pi = \pi_g$ is a differential operator
Generator	$X_H(x) = \sum_s P^{sj}(x) \frac{\partial H}{\partial x^s}(x)$	$A_h := \nabla^i \nabla_j h$
Dynamics	$\frac{d}{dt} x(t) = X_H(x(t))$. It is a system of n ODEs	$\frac{\partial u(t,x)}{\partial t} = A_h \frac{\partial u(t,x)}{\partial x}$. It is a system of n PDEs

- The PDE-system $\frac{\partial u(t,x)}{\partial t} = A_h \frac{\partial u(t,x)}{\partial x}$ is geometric: if we change the coordinate system on the manifold, A_h changes as (1,1)-tensor.
- The condition that the metric g is flat is the (∞-dimensional version of the) Jacobi identity.
- Interpretation of the PDE $\frac{\partial u(t,x)}{\partial x} = A_h \frac{\partial u(t,x)}{\partial x}$ as a Hamiltonian flow is nontrivial; I will speak about it on the next slide. But it is very useful: many facts known from the theory of finite-dimensional Hamiltonian systems have their analogs in the ∞-dimensional setting. Physicists (e.g. Landau) explored this already in 40th; mathematicians came in this science in 1980th (e.g., Gelfand & Dorfman, Magri & Morosi).

Let us compare the ODE $\frac{d}{dt}x(t) = X_H$ with the PDE $\frac{\partial u(t,x)}{\partial t} = A_h \frac{\partial u(t,x)}{\partial x}$

- The PDE $\frac{\partial u(t,x)}{\partial t} = A_h \frac{\partial u(t,x)}{\partial x}$ is of Cauchy-Kovalewskaya type: the t-derivatives are expressed in terms of x-derivatives.
- For initial real-analytic data $\hat{u}(x) = u(t_0, x)$ there exists a unique local solution $u(t, x)$.
- Real-analyticity is important in the case when the matrix A has complex eigenvalues; in this case smoothness is not sufficient for the existence of solution.
- We will work in the real-analytic category; we may work locally (so "points" are real-analytic curves $u(x)$).

∞-dimensional Poisson brackets of the first order

We defined, visually artificially, the "Hamiltonian" operator as follows: by a function $h : M^n \to \mathbb{R}$ it constructs the (1,1)-tensor $\nabla^i \nabla_j h \frac{\partial u^j}{\partial x}$. Let us explain why we consider the formula as an analog of $\frac{d}{dt}x = X_H$, and generalise it on the case when h is a $\underbrace{\text{differential polynomial}}_{\text{to be explained}}$.

Our "point" is a curve $u(x)$. Our solution $u(t, x)$ will describe the time-evolution of points. A "tangent vector" to a "point" is a vector field $v(x)$ along the curve.

Our Hamiltonian is the functional $u(x) \mapsto \int_{S^1} h(u(x))dx \in \mathbb{R}$.
Its "differential" is the variational derivative; on the vector field $v^i(x)$ ("tangent vector") it takes the value $\int_{S^1} v^i \frac{\delta h}{\delta u^i} dx$.

In our simplest situation, when h is just a function of u, $\frac{\delta h}{\delta u^i} = \frac{\partial h}{\partial u^i}$.

Poisson bracket, local and of the first order, sends, at the "point" $u(t)$, the pair of functionals generated by functions h and f, to the functional

$$\left\{ \int_{S^1} f dx, \int_{S^1} h dx \right\} = \int \frac{\delta f}{\delta u^i} \left(g^{ij} \frac{d}{dx} - \Gamma^{ij}_s \frac{\partial u^s}{\partial x} \right) \frac{\delta h}{\delta u^j} dx.$$

The formula is independent on the local coordinate system u on our manifold M; the pair (g, Γ) transforms as it should.
In particular, we have $\left(g^{ij} \frac{d}{dx} - \Gamma^{ij}_s \frac{\partial u^s}{\partial x} \right) \frac{\delta h}{\delta u^j} = \underbrace{\nabla^i \nabla_j h}_{A} \frac{\partial u^j}{\partial x}$ as in my "table".

The formulas on the previous slide are well adapted to the case when the functional $u(x) \mapsto \int_{S^1} h(u(x))dx$ is more complicated: instead of the function $h(u)$ we may consider functions $h(u, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \ldots)$ on the jet space of the curves (it is sufficient to consider those which are polynomial in $\frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \ldots$; they are called differential polynomials).

Indeed, the formula is

$$\left\{ \int_{S^1} f dx, \int_{S^1} h dx \right\} = \int \frac{\delta f}{\delta u^i} \left(g^{ij} \frac{d}{dx} - \Gamma^{ij}_s \frac{\partial u^s}{\partial x} \right) \frac{\delta h}{\delta u^j} dx$$

and one apply it to $h(u, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \ldots)$ and $f(u, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \ldots)$; in this case

$$\frac{\delta h}{\delta u^i} = \sum_s (-1)^s \frac{d^s}{dx^s} \frac{\partial h}{\partial u^i_{x^s}}.$$

Jacobi-identity

Fact (Dubrovin, Novikov, 1989). The bilinear skew-symmetric bracket

$$\left\{\int_{S^1} f dx, \int_{S^1} h dx\right\} := \int_{S^1} \tfrac{\delta f}{\delta u^i}\left(g^{ij}\tfrac{d}{dx} - \Gamma^{ij}_s \tfrac{\partial u^s}{\partial x}\right)\tfrac{\delta h}{\delta u^j} dx$$

(with g^{ij} a contravariant metric and Γ^{ij}_s its Christoffel symbols with raised index) satisfies the Jacobi identity if and only if g is flat.

The direction "$\Leftarrow$" is almost trivial: by construction the formula does not depend on the choice of coordinates, so one can check the statement in the coordinates where g^{ij} are constants and $\Gamma^{ij}_s = 0$. The direction "$\Rightarrow$" can be obtained by direct calculations (alternatively, one can employ the representation theory and use that Jacobi condition should be a geometric condition on the metric and its first and second derivatives so it should be invariantly expressed in the terms of curvature).

In the 4th Lecture we (actually, A.B.) will discuss also Poisson structures of the **third** order. They will be given by a similar formula:

$$\left\{\int_{S^1} f dx, \int_{S^1} h dx\right\} := \int_{S^1} \tfrac{\delta f}{\delta u^i}\left(a^{ij}\tfrac{d^3}{dx^3}\tfrac{\delta h}{\delta u^j} + \ldots\right) dx.$$

Conservative quantities and commuting fields

- In the finite-dimensional case, conservative quantity are usually called integrals; they are defined as follows:
  - A function $f : M \to \mathbb{R}$ is an integral of a Hamiltonian system on $(M, \{\ ,\ \} = P^{ij})$, if $\{H, F\} = 0$. This property is equivalent to the property that F is constant one the orbits of X_H.
  - If F is integral, then the vector field $X_F := P^{si} \frac{\partial F}{\partial x^s}$ commutes with the vector field X_H; so the flows of these vector fields commute: if we denote the time of the flow of X_F by τ, then

    $$\Psi_\tau^{X_F} \circ \Psi_t^{X_H} = \Psi_t^{X_H} \circ \Psi_\tau^{X_F}.$$

    Commutativity follows directly from the Jacobi identity.
- In our infinite-dimensional case, conservative quantity (of a system generated by h) is a function (or differential polynomial) f such that $\{\int_{S^1} h dx, \int_{S^1} f dx\} = 0$. Also in this case the flows commute: if we denote the t-parameter of the second system by τ, for any initial data $u(t_0, \tau_0, x)$ there exists a local solution $u(t, \tau, x)$ satisfying simultaneously both PDE-equations:

  $$\frac{\partial u^i}{\partial t} = \nabla^i \nabla_s h \frac{\partial u^s}{\partial x} \quad , \quad \frac{\partial u^i}{\partial \tau} = \nabla^i \nabla_s f \frac{\partial u^s}{\partial x}.$$

	finite dimension	infinite dimension
Integral	a function F constant on solutions	a functional $\int_{S^1} f(x(\tau))d\tau$ constant on solutions.

- In the finite-dimensional case it is known that if F is integral, then the vector field X_F commutes with X_H.This allows to reduce the degree of the freedom by 2 ("symplectic reduction"):
  - first part of the reduction is restricting everything to $F = \text{const}$.
  - The second part of reduction uses $[X_F, X_H] = 0$.
  - In particular, if a system on $2n-$dimensional (compact) symplectic manifold admits n independent integrals, one can integrate it in quadratures (Arnold-Liouville Theorem)
- In the infinite-dimensional case, if $\int_{S^1} f(u(x))dx$ is an integral, then the flow generating by the operator $A_f = \nabla^i \nabla_j f$ commutes with that of A_h. That is, for any initial data $u(t_0, \tau_0, x)$ there exists a (local) function $u(t, \tau, x)$ such that

$$\frac{\partial u(t,\tau,x)}{\partial t} = A_h \frac{\partial u(t,\tau,x)}{\partial x} \quad ; \quad \frac{\partial x(t,\tau,x)}{\partial \tau} = A_f \frac{\partial u(t,\tau,x)}{\partial x}.$$

- In the infinite-dimensional case, it allows to find certain solutions; moreover, it allows to reduce the initial nonlinear equations $\frac{\partial u(t,x)}{\partial t} = A_h \frac{\partial u(t,x)}{\partial x}$ to certain linear PDE of the same type (method of Tsarev).

Bi- and multihamiltonian structures

	finite dimension	infinite dimension
Background	Poisson manifold (M^n, P^{ij})	flat (M^n, g)

Def. Two Poisson structures are compatible, if the sum is also a Poisson structure. In this case, any their linear combination is a Poisson structure

- The sum of two Poisson structures is bilinear and skew-symmetric. Thus, the only condition is that the Jacobi identity is fulfilled
  - In finite dimension, it is a nonlinear condition on P^{ij} and $\bar{P}^{ij}$. Under some nondegeneracy assumption, this condition is equivalent to the condition that the operator L such that $\sum_s L^i_s P^{sj} = \bar{P}^{ij}$ is Nijenhuis; though compatible Poisson structures play an important role in the finite dimensional integrable systems, and clearly have a relation to Nijenhuis geometry, in our lectures we do not discuss finite dimensional case.
  - In the ∞-dimensional case the compatibility condition reads as follows and will be explained in the next slide: if the first Poisson structure is given by flat g and the second by flat $\bar{g}$, then the compatibility condition is the following two conditions:
    - $L = g^{-1}\bar{g}$ is Nijenhuis,
    - $g^{ij} + \bar{g}^{ij}$ is flat. If L has n different eigenvalues, first condition implies the second (Ferapontov 1991/ Mokhov 1998).

Compatible Poisson structures in the $\infty-$dimensional case

Recall that Poisson structure of the first order is generated by a flat metric g and is given, at a "point" $u(x)$ by the following formula:

$$\left\{\int_{S^1} f dx, \int_{S^1} h dx\right\} := \int_{S^1} \frac{\delta f}{\delta u^i}\left(g^{ij}\frac{d}{dx} - \Gamma^{ij}_s \frac{\partial u^s}{\partial x}\right)\frac{\delta h}{\delta u^j} dx.$$

The sum of two such structures is given by

$$\left\{\int_{S^1} f dx, \int_{S^1} h dx\right\} := \int_{S^1} \frac{\delta f}{\delta u^i}\left((g^{ij} + \bar g^{ij})\frac{d}{dx} - (\Gamma^{ij}_s + \bar\Gamma^{ij}_s)\frac{\partial u^s}{\partial x}\right)\frac{\delta h}{\delta u^j} dx. \qquad (2)$$

Next, as we know the condition that (2) is skew-symmetric and satisfied the Jacobi identity is that $g^{ij} + \bar g^{ij}$ is a flat metric and $(\Gamma^{ij}_s + \bar\Gamma^{ij}_s)$ is its Christoffel symbols with one index lifted.

Fact (Ferapontov/Mokhov 1991): If two contravariant metrics g and $\bar g$ have the property that for their sum $\hat g = g + \bar g$ we have

$$\hat\Gamma^{ij}_s = \Gamma^{ij}_s + \bar\Gamma^{ij}_s$$

then the operator $L^i_j = \bar g^{is} g_{js}$ is Nijenhuis.

Moreover, if $L^i_j = \bar g^{is} g_{js}$ is Nijenhuis and has $n = \dim\ M$ different eigenvalues, then statement is true in the other direction.

Moreover, if L is Nijenhuis and has n different eigenvalues, and if g and $\bar g$ is flat, then also $\hat g = g + \bar g$ is flat.

Proof can be done by direct calculations. At the end of the lecture I give a proof under the assumptions that L has n different eigenvalues.

Thus, the compatibility of the Poisson structures corresponding to g and $\bar{g}$ is reduced to the following geometric problem:

- $L^i_j = \bar{g}^{is} g_{js}$ is Nijenhuis and has n different eigenvalues.
- g and $\bar{g}$ are flat

We see that the geometric objects (g, L) to which study the problem is reduced to are the same as in projectively equivalent metrics.

It is not a coincidence:

Sinjukov-Topalov hijerarchy of projectively equivalent metrics

Theorem 1. Let g^{ij} and L^i_j are projectively compatible. Then, $Lg := L^i_s g^{sj}$ and L are also projectively compatible. Moreover, if g is flat then Lg has constant, possibly zero curvature. The curvature of g is zero if and only if $\Delta_g(\operatorname{trace} L) = 0$.
Moreover, if the metric $\bar{g} = Lg$ is flat, then the metric is Poisson compatible to g, in the sense that $\hat{g} = g + \bar{g}$ is flat and we have $\hat{\Gamma}^{ij}_s = \Gamma^{ij}_s + \bar{\Gamma}^{ij}_s$.

Authorships: The first statement is due to Sinjukov 1966. Topalov 1998 (and later Bolsinov-Matveev 2006) extended the statement replacing L first by polynomial of L and then by arbitrary function of L. The statement that Lg has constant curvature may be new (Bolsinov-Konyaev-Matveev 2020). The statement is that the metrics are Poisson-compatible is in (Bolsinov-Konyaev-Matveev 2020))

Including metrics of constant curvature

Metrics of constant curvature K allow to construct $\infty-$dimensional Poisson structures (Ferapontov-Pavlov 1991): the formula is:

$$\left\{\int_{S^1} f dx, \int_{S^1} h dx\right\} := \int_{S^1} \frac{\delta f}{\delta u^i}\left(g^{ij}\frac{d}{dx} - \Gamma^{ij}_s \frac{\partial u^s}{\partial x} + \frac{\partial u^i}{\partial x} K \left(\frac{d}{dx}\right)^{-1} \frac{\partial u^j}{\partial x}\right)\frac{\delta h}{\delta u^j} dx.$$

Such Poisson structures are called weakly nonlocal.

I will not give a mathematically clear explanation of what is the inverse $\left(\frac{d}{dx}\right)^{-1}$ of $\frac{d}{dx}$ but if h is a function, $\frac{\delta h}{\delta u^j}$ is just $\frac{\partial h}{\partial u^j}$ and $\left(\frac{d}{dx}\right)^{-1}\left(\frac{\partial u^j}{\partial x}\frac{\partial h}{\partial u^j}\right) = h$.

In particular the Hamiltonian flow is generated by the operator

$$A_h := \nabla^i \nabla_j h + K h \delta^i_j.$$

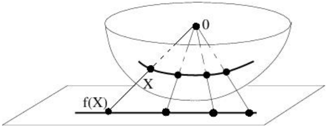
The formula clearly generalised the formula in the flat case, when $K = 0$.

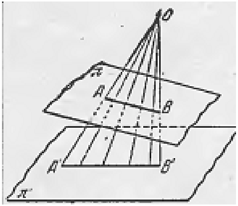
Metrics of constant curvature can be included in the Theorem above.

Theorem 1b. Let g^{ij} be a flat metric. The following statements hold:

1. If L is geodesically compatible with g and non-degenerate, then the metric Lg has constant (possibly zero) curvature and is Poisson compatible with g.
2. If non-degenerate L_1 and L_2 are geodesically compatible with g, then the metrics $L_1 g$ and $L_2 g$ are Poisson-compatible.

At the beginning of my talk I recalled that projectively equivalent metrics of constant curvature are completely understood; see also Lecture 1:





This allows to describe all Poisson-compatible metrics arising from geodesic equivalence.

Theorem 2. Let g be a flat metric and $x^1, \ldots, x^n$ be local coordinates in which all the components of g are constant (i.e., local flat coordinates).
Suppose g and $\bar{g}$ are geodesically equivalent, consider $L^{ij} = \bar{g}^{ij} \cdot \left| \frac{\det \bar{g}}{\det g} \right|^{\frac{1}{n+1}}$ (it is the same L as before, with the index raised by g). Then, it is given by the formula

$$L^{ij} = a^{ij} + b^i x^j + b^j x^i - K x^i x^j. \qquad (3)$$

Here (a^{ij}, b^i, K) are constants and $a^{ij} = a^{ji}$.
Moreover, near those points where L^{ij} is non-degenerate and, therefore, defines a pseudo-Riemannian contravariant metric, the curvature of this metric is constant and equals K.

We thus obtained a huge family of Poisson-compatible metrics. This family contains examples known before, and also new examples.

Old and new examples

Let me give one example of flat projectively equivalent metrics $g \overset{g.e.}{\sim} \bar{g}$. From Theorem 2 we know that the parameters are $g^{ij}, a^{ij} \in \mathrm{SymMat}_{\mathbb{R}}(n \times n)$, $b \in \mathbb{R}^n$, $K \in \mathbb{R}$.
Example 1 (Ferapontov – Pavlov 1991/ Antonowicz – Fordy 1987).

$$g = \begin{pmatrix} & & & 1 \\ & & \cdot^{\cdot^{\cdot}} & \\ & \cdot^{\cdot^{\cdot}} & & \\ 1 & & & \end{pmatrix}, \quad a = \begin{pmatrix} & & & 0 \\ & & \cdot^{\cdot^{\cdot}} & 1 \\ & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \\ 0 & 1 & & \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad K = 0.$$

This pair (g, L) is much better known in the coordinates such that L is diagonal; in this case it is just the Levi-Civita formula we obtained in Lecture 1:

$$L^i_j = \mathrm{diag}(x^1, x^2, ..., x^n)\ , \quad g = \sum_i \left(\prod_{s \neq i} (x^i - x^s) \right) (dx^i)^2.$$

For example in dimension 3:

$$g = (x^1 - x^2)(x^1 - x^3)(dx^1)^2 - (x^2 - x^1)(x^2 - x^3)(dx^2)^2 + (x^3 - x^1)(x^3 - x^2)(dx^3)^2.$$

Example 2.

$$g_{ij} = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{pmatrix}, \ A = \begin{pmatrix} & & & 0 \\ & & 0 & 1 \\ & 0 & 1 & \\ 0 & 1 & & \end{pmatrix}, \ b = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \ K = 0$$

In this case

$$L^{ij} = \begin{pmatrix} 0 & 0 & x^1 & 0 \\ 0 & 0 & x^2 & 1 \\ x^1 & x^2 & 2\,x^3 + 1 & x^4 \\ 0 & 1 & x^4 & 0 \end{pmatrix}.$$

The operator L^i_j has two Jordan 2×2 blocks with nonconstant different eigenvalues. After a suitable coordinate transformation, g and L become

$$g_{ij} = \begin{pmatrix} 2\left(-2\,x^2 - 1\right)\left(x^3 - x^1\right) & \left(x^3 - x^1\right)^2 & 0 & 0 \\ \left(x^3 - x^1\right)^2 & 0 & 0 & 0 \\ 0 & 0 & 2\left(-2\,x^4 - 1\right)\left(x^1 - x^3\right) & \left(x^1 - x^3\right)^2 \\ 0 & 0 & \left(x^1 - x^3\right)^2 & 0 \end{pmatrix},$$

$$L^i_j = \begin{pmatrix} x^1 & 0 & 0 & 0 \\ 2\,x^2 + 1 & x^1 & 0 & 0 \\ 0 & 0 & x^3 & 0 \\ 0 & 0 & 2\,x^4 + 1 & x^3 \end{pmatrix}.$$

The example is new and interesting because of the following: In the theory of integrable systems of hydrodynamic type one customary assumes that the operators defining these systems are diagonalisable and this special case is well studied. The case of other Segre characteristics is considered to be much harder and there are only very few such examples in the literature. We may deliver many such examples.

One more construction coming from Theorem 1:

Theorem 1b. Let g^{ij} be a flat metric. The following statements hold:

1. If L is projectively compatible with g and non-degenerate, then the metric Lg has constant (possibly zero) curvature and is Poisson compatible with g.
2. If non-degenerate L_1 and L_2 are geodesically compatible with g, then the metrics L_1g and L_2g are Poisson-compatible.

Let us start from a projectively compatible (g, L) such that g is flat. We know that Lg has constant curvature and (Lg, L) is projectively compatible. If Lg is flat, we can continue further: we can take (L^2g, L); it is again geodesically compatible and of constant curvature. If it is flat, we can take (L^3g, L). We can repeat the procedure until the step k such that (L^kg, L) is of constant curvature and is not flat (in the next step the pair $(L^{k+1}g, L)$ is still projectively compatible but is not of constant curvature).

All metrics we consider, and all their linear combinations, are Poisson compatible: for any constants $a_0, ..., a_k$ the polynomial family $(a_0\mathrm{Id} + a_1L + ... + a_kL^k)g$ is a Poisson compatible pencil.

Poisson-compatible pencil for two examples

We consider the Levi-Civita metric: it is diagonal and in contravariant form it given by

$$g^{ii} = \frac{\alpha_i(x^i)}{\prod_{j\neq i}(x^i - x^j)} \quad , L = \mathrm{diag}(x^1, ..., x^n)$$

In this case, for $\alpha_i = (-1)^i$ we precisely obtain the metric from Example 1. Next, in this case $k = n + 1$; so for any $a_0, ..., a_{n+1} \in \mathbb{R}$ the family $(a_0 \mathrm{Id} + a_1 L + ... + a_{n+1} L^{n+1})g$ is Poisson-compatible.

Example 2.

$$g_{ij} = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{pmatrix}, \; A = \begin{pmatrix} & & & 0 \\ & & 0 & 1 \\ & 0 & 1 & \\ 0 & 1 & & \end{pmatrix}, \; b = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \; K = 0$$

(The example exists in any even dimension $2n$; the only nonzero element of b should stay on place $n+1$).
In this case, for any polynomial of any degree k the family $(a_0 \mathrm{Id} + a_1 L + ... + a_k L^k) g$ is Poisson-compatible.

Integrable systems generated by compatible Poisson structures

Def. A function F (or a functional $\int_{S^1} f dx$) is a Casimir for a Poisson structure, if $\{F,\ .\} = 0$.

- In the finite-dimensional case, this simply means $P^{is}\frac{\partial F}{\partial x^i} = 0$.
- In the case when f is just a function, this means $\nabla_i \nabla_j f = 0$ in the flat case and $\nabla_i \nabla_j f + Kf = 0$ in the constant curvature K case.

Fact (Folklore; well-known in finite dimension; e.g., Dubrovin and Magri in infinite dimension): Suppose Poisson structures $\{\ ,\ \}_1$ and $\{\ ,\ \}_2$ are compatible. Then, for two Casimirs K_1 and K_2 of $\{\ ,\ \}_2$ commute with respect to $\{\ ,\ ,\}_1$, that is, $\{K_1, K_2\}_1 = 0$.
In particular, they generate commutative flows.

What are Casimirs for functions in our examples

Theorem 3. Aussume g is flat and (g, L) is projectively compatible. Then, the function $f = \sqrt{\det(L)}$ is a Casimir of LgL. The corresponding A_f, with respect to any of the Poisson structures from the pencil, is given by

$$A_f = \text{const} \cdot \frac{1}{\sqrt{\det L}} L^{-1}.$$

Corollary. Let (g, L) be projectively compatible and L has $n = \dim M$ different eigenvalues. Then, for any α the Hamiltonian system generated by $h_\alpha = \sqrt{\det(L - \alpha \mathrm{Id})}$ is integrable in the Tsarev sense.

Example. Consider the projectively compatible (g, L) from Example 1:

$$L^i_j = \mathrm{diag}(u^1, u^2, ..., u^n) \ , \quad g = \sum_i \left(\prod_{s \neq i} (u^i - u^s) \right) (du^i)^2.$$

Then, the integrable PDE coming from Theorem 3 is

$$\frac{\partial u}{\partial t} = \frac{1}{\sqrt{\det(L + \alpha \mathrm{Id})}} (L + \alpha \mathrm{Id})^{-1} \frac{\partial u}{\partial x}.$$

The integrals are $\sqrt{\det (L + \beta \mathrm{Id})}$ and the commuting flows are given by the PDE

$$\frac{\partial u}{\partial \tau} = \frac{1}{\sqrt{\det(L + \beta \mathrm{Id})}} (L + \beta \mathrm{Id})^{-1} \frac{\partial u}{\partial x}.$$

How special is the Hamiltonian system coming from projective equivalence?

- There are much more Hamiltonian systems of hydrodynamic type than that coming from projective equivalence
- In Lecture 4 we will show that if we consider compatible Poisson structures which are sums of 1st and 3rd order Poisson structures, then "most" such Poisson pensils come from projective equivalence.
  - There is also a direct relation to separation of variables
- The next result shows, that the Hamiltonian systems of hydrodynamic type than those that come from projective equivalence is in certain sense "maximally superintegrable"

Theorem 4. Let L be a Nijenhuis operator which is invertible, self-adjoint with respect to a metric g and differentially non-degenerate almost everywhere. Suppose the metrics gL^{-k} are flat for $k = 0, \ldots, n$ and the metric gL^{-n-1} has constant curvature $K \neq 0$. Then g and L are geodesically compatible. Moreover, in a neighborhood of every point, where L is differentially non-degenerate, the pair (g, L) is locally isomorphic (up to multiplication of g by a constant) to the one coming from example 1.

I will explain the proof and within the proof explain also the result (of Mokhov) which I claimed in the first part: if g is flat, L is Nijenhuis with n different eigenvalues and Lg is flat, then any linear combination of g and Lg is also flat.

The proof is actually by calculations

Suppose L is differentially-non-degenerate, then at almost every point it has n different eigenvalues. We will work in a small neighbourhood of such a point and first consider the case when the eigenvalues are real. In this case, there exists a coordinate system $x^1, \ldots, x^n$ such that L is given by

$$\operatorname{diag}\left(x^1, x^2, \ldots, x^n\right). \tag{4}$$

In this coordinates the metric g is also diagonal, $g = \mathrm{diag}(g_1, ..., g_n)$.
We will proceed as follows: we will calculate the Cristoffel symbols Γ^i_{jk} and $\bar\Gamma^i_{jk}$ of the metrics g and $\bar g := P(L)g$. Then, we calculate the components of the curvature $R^{ij}{}_{k\ell}$ and $\bar R^{ij}{}_{k\ell}$; equating them to zero will give **linear** equations on $P(x^i)$ which will give us the only solution.

First step: standard “3rd year” calculations

Lemma.
Then the Christoffel symbols Γ^i_{jk} of the metric g are as follows:

- $\Gamma^k_{ij} = 0$ for pairwise different i, j and k,
- $\Gamma^k_{kj} = \frac{1}{2}\frac{1}{g_k}\frac{\partial g_k}{\partial x^j}$ for arbitrary k, j,
- $\Gamma^k_{jj} = -\frac{1}{2}\frac{1}{g_k}\frac{\partial g_j}{\partial x^k}$ for arbitrary $k \neq j$.

Consequently, the Christoffel symbols $\bar{\Gamma}^i_{jk}$ of the metric $\bar{g} = Lg$ are as follows:

- $\bar{\Gamma}^k_{ij} = 0$ for pairwise different i, j and k,
- $\bar{\Gamma}^k_{kj} = \Gamma^k_{kj}$ for arbitrary $k \neq j$,
- $\bar{\Gamma}^k_{jj} = \frac{h_k}{h_j}\Gamma^k_{jj}$ for arbitrary $k \neq j$,
- $\bar{\Gamma}^k_{kk} = \Gamma^k_{kk} - \frac{1}{2}\frac{1}{h_k}h'_k$ for arbitrary k,

(each h_k is a function of one variable, so h'_k in the latter formula and below, in e.g. (5), is just the usual derivative, $h'_k = \frac{\partial h_k}{\partial x^k}$).

Curvatures of the metrics g and $\bar{g}$

Lemma. The components of the curvature tensors ${R^i}_{jk\ell}$ and ${\bar{R}^i}_{jk\ell}$ of the connections Γ and $\bar{\Gamma}$ are as follows: (no summation over repeating indices)

- ${R^s}_{ijk} = 0$ for arbitrary pairwise different i, j, k and s,
- ${R^i}_{ijk} = 0$ for arbitrary i, j, k,
- ${R^j}_{ijk} = -\frac{\partial \Gamma^j_{ij}}{\partial x^k} + \Gamma^j_{ji}\Gamma^i_{ik} + \Gamma^j_{jk}\Gamma^k_{ik} - \Gamma^j_{kj}\Gamma^j_{ij}$ for arbitrary $j \neq i$ and $i \neq k$,
- ${R^j}_{iji} = \frac{\partial \Gamma^j_{ii}}{\partial x^j} - \frac{\partial \Gamma^j_{ij}}{\partial x^i} + \sum_{\alpha=1}^{n} \Gamma^j_{j\alpha}\Gamma^\alpha_{ii} - \Gamma^j_{ii}\Gamma^i_{ij} - \Gamma^i_{ij}\Gamma^i_{ij}$ for arbitrary i and j,
- ${R^k}_{iji} = -{R^k}_{iij} = \frac{\partial \Gamma^k_{ii}}{\partial x^j} + \Gamma^k_{jj}\Gamma^j_{ii} + \Gamma^k_{jk}\Gamma^k_{ii} - \Gamma^k_{ii}\Gamma^i_{ij}$ for arbitrary $k \neq i$ and $k \neq j$.
- ${\bar{R}^s}_{ijk} = 0$ for arbitrary pairwise different i, j, k and s,
- ${\bar{R}^i}_{ijk} = 0$ for arbitrary i, j, k,
- ${\bar{R}^j}_{ijk} = {R^j}_{ijk}$ for arbitrary $j \neq i$ and $i \neq k$.
- For arbitrary $i \neq j$,

$$
{\bar{R}^j}_{iji} = \frac{h_j}{h_i}\frac{\partial \Gamma^j_{ii}}{\partial x^j} - \frac{\partial \Gamma^j_{ij}}{\partial x^i} + \sum_{\alpha=1}^{n} \frac{h_\alpha}{h_i}\Gamma^j_{j\alpha}\Gamma^\alpha_{ii} - \frac{h_j}{h_i}\Gamma^j_{ii}\Gamma^i_{ij} - \Gamma^j_{ij}\Gamma^j_{ij} - \frac{1}{2}\frac{h'_i}{h_i}\Gamma^j_{ij} + \frac{1}{2h_i}h'_j\Gamma^j_{ii}\,, \quad (5)
$$

- ${\bar{R}^k}_{iji} = \frac{h_k}{h_i}{R^k}_{iji}$.

We assumed that g is flat and $\bar{g}$ is flat for $h(x^i)$ which is polynomial of degree $\leq n$ and is constant otherwise. Then, the only component which can be nonzero is:

$$\bar{R}^{ij}{}_{ij} = \frac{h_j}{g_i}\frac{\partial \Gamma^j_{ii}}{\partial x^j} - \frac{h_i}{g_i}\frac{\partial \Gamma^j_{ij}}{\partial x^i} + \sum_{\alpha=1}^{n}\frac{h_\alpha}{g_i}\Gamma^i_{j\alpha}\Gamma^\alpha_{ii} - \frac{h_j}{g_i}\Gamma^j_{ii}\Gamma^i_{ij} - \frac{h_i}{g_i}\Gamma^j_{ij}\Gamma^j_{ij} - \frac{h'_i}{2g_i}\Gamma^j_{ij} + \frac{h'_j}{2g_i}\Gamma^j_{ii}.$$

We see that the equation is linear in h_i, which proved the result of Mokhov
By our assumptions the component $\bar{R}^{ij}{}_{ij}$ vanished for $h_i(x^i) = P(x^i)$ for polynomial of degree n and is equal to constant for polynomial of degree $n+1$. This gives us the following system of linear equations on Γ^i_{jk} and g_i:

$$\frac{P(x^j)}{g_i}\frac{\partial \Gamma^j_{ii}}{\partial x^j} - \frac{P(x^i)}{g_i}\frac{\partial \Gamma^j_{ij}}{\partial x^i} + \sum_{\alpha=1}^{n}\frac{P(x^\alpha)}{g_i}\Gamma^j_{j\alpha}\Gamma^\alpha_{ii} - \frac{P(x^j)}{g_i}\Gamma^j_{ii}\Gamma^i_{ij}$$
$$- \frac{P(x^i)}{g_i}\Gamma^j_{ij}\Gamma^j_{ij} - \frac{P'(x^i)}{2g_i}\Gamma^j_{ij} + \frac{P'(x^j)}{2g_i}\Gamma^j_{ii} = a_{n+1}K.$$

The solution gives precisely the Levi-Civita formula for the metrics.