Nijenhuis Geometry: gl-regular operators, quasilinear PDEs and analytic matrix functions

Alexey Bolsinov and Vladimir Matveev

Lecture 2

MATRIX-SMRI Symposium:
Nijenhuis Geometry and Integrable Systems
Sydney, Australia
February, 2022

Outline of Lecture 2

- Terminology: singular and generic points.
- gl-regular Nijenhuis operators
  - What is a regular/typical operator?
  - gl-regular operators and their properties
  - Diagonalisation vs companion form
  - First and second companion forms for Nijenhuis operators
  - Magic formula
- Quasilinear systems of PDEs of hydrodynamic type:
  - Conservation laws
  - Commuting flows, symmetries
  - Integrability
  - Special case of Nijenhuis operators
- Real-analytic functions of Nijenhuis operators
  - What is an analytic matrix function?
  - Canonical complex structure associated with Nijenhuis operator
  - Generalised Nirenberg-Newlander theorem
  - Some global applications

Generic and singular points

Definition

- A point $p \in M$ is called *algebraically generic*, if the algebraic type of L does not change in some neighbourhood $U(p) \subset M$.
- A point $p \in M$ is called *singular*, if it is not algebraically generic.
- A point $p \in M$ is called *differentially non-degenerate*, if the differentials $\mathrm{d}\sigma_1(x), \ldots, \mathrm{d}\sigma_n(x)$ of the coefficients of the characteristic polynomial of $L(x)$ are linearly independent at this point.
- A point $p \in M$ is called *of scalar type*, if $L(p) = \lambda \cdot \mathrm{Id}$, $\lambda \in \mathbb{R}$.
- An operator $L(p)$ (and the corresponding point $p \in M$) is called gl-*regular*, if its $\mathrm{GL}(n)$-orbit $\mathcal{O}\big(L(p)\big) = \{X\, L(p)\, X^{-1} \mid X \in \mathrm{GL}(n)\}$ has maximal dimension, namely, $\dim O\big(L(p)\big) = n^2 - n$.

What is a *regular* linear operator?

Natural requirements:

- set of "regular objects" must be open and dense
- condition of being "regular" must be invariant, algebraic and easy to check

In Riemannin and symplectic geometries:

$$\det(g_{ij}) \neq 0 \quad \text{и} \quad \det(\omega_{ij}) \neq 0.$$

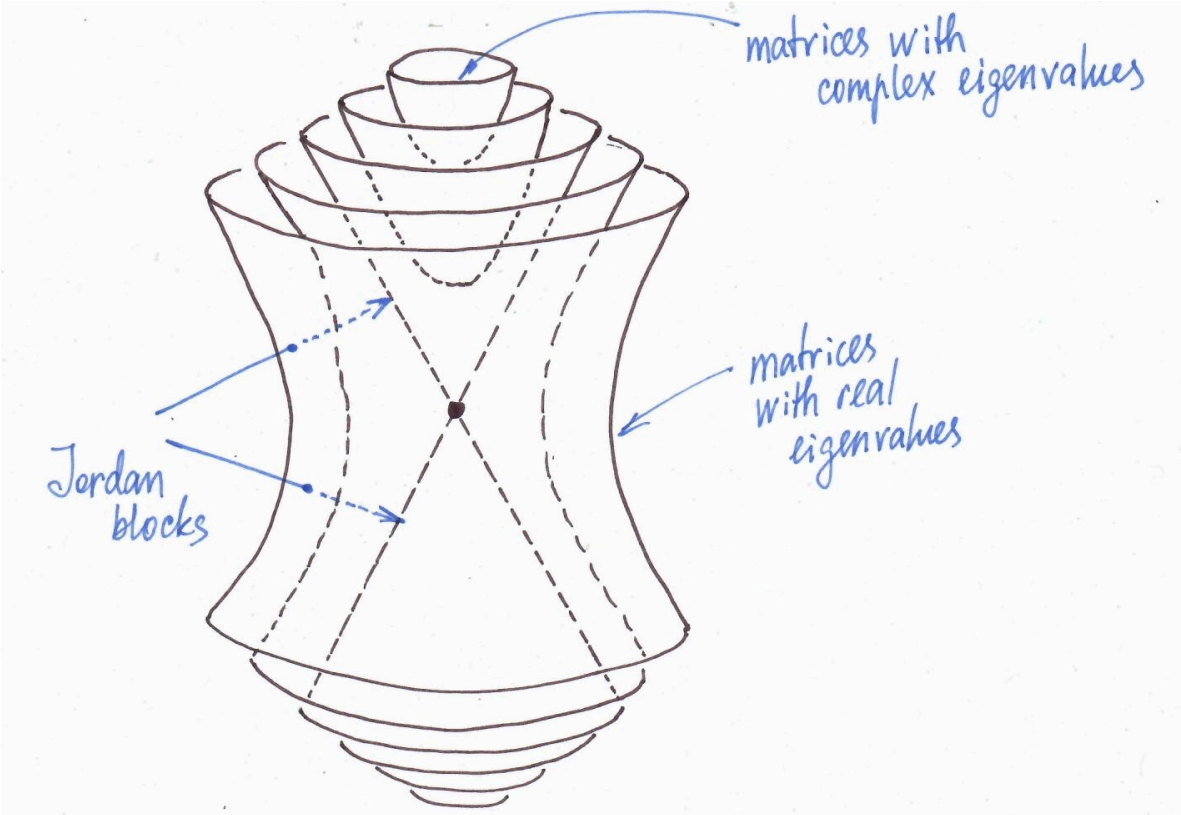
In the case of operators, $\det L \neq 0$ makes sense, but not very natural...

The invariance condition says that "being regular" is related to the natural action of $\mathrm{GL}(n, \mathbb{R})$ (group of coordinate transformations):

$$L \mapsto ALA^{-1}, \qquad A \in \mathrm{GL}(n, \mathbb{R}).$$

In other words, "regular" is a condition not on the operator itself, but on its orbit. For orbits, the most natural regularity condition is to be of maximal dimension. That is what we are going to do.

Types (orbits) of traceless matrices/operators



Partition of $sl(2,\mathbb{R})$, space of traceless operators, into orbits is defined by

$$\det \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = -a^2 - bc = \text{const}.$$

"Good" matrices have 2-dimensional orbits. The only "bad" matrix is the zero matrix.

What is a gl-regular operator (matrix)?

Definition

A linear operator $L : \mathbb{R}^n \to \mathbb{R}^n$ is called gl-regular, if one of the following equivalent conditions hold

- the dimension of its orbit $\mathcal{O}(L) = \{ALA^{-1} \mid A \in \mathrm{GL}(n, \mathbb{R})\}$ is maximal (and equals $n^2 - n$)
- $\mathrm{Id}, L, L^2, \ldots, L^{n-1}$ linearly independent
- Jordan normal form of L contains exactly one Jordan block for each eigenvalue
- there exist a vector ξ such that $\xi, L\xi, \ldots, L^{n-1}\xi$ form a basis
- the coefficients $\sigma_1, \ldots, \sigma_n$ of the characteristic polynomial (as functions on $\mathrm{gl}(n)$) are functionally independent at $L \in \mathrm{gl}(n)$
- the functions $\tau_1 = \operatorname{tr} L, \tau_2 = \operatorname{tr} L^2, \ldots, \tau_n = \operatorname{tr} L^n$ on $\mathrm{gl}(n)$ are functionally independent at $L \in \mathrm{gl}(n)$

What is a gl-regular operator (continued...)?

- L can be reduced to the form

$$L_{\text{comp1}} = \begin{pmatrix} \sigma_1 & 1 & \dots & 0 \\ \vdots & & \ddots & \\ \sigma_{n-1} & & & 1 \\ \sigma_n & 0 & \dots & 0 \end{pmatrix}$$

- as well as to

$$L_{\text{comp2}} = \begin{pmatrix} 0 & 1 & \dots & 0 \\ \vdots & & \ddots & \\ 0 & & & 1 \\ \sigma_n & \sigma_{n-1} & \dots & \sigma_1 \end{pmatrix}$$

Important property. The complement to the set of gl-regular operators has codimension 3.

Diagonalisation vs companion form

Given a matrix L of a linear operator, we want to change a basis to reduce L to a simpler (canonical form).
The standard approach is to (try to) diagonalise L by means of a suitable similarity transformation:

$$L \mapsto P^{-1} L\ P = D = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}.$$

If diagonalisation is impossible, then D is "replaced" by Jordan normal form.
For gl-regular matrices, we can use another canonical form

$$L \mapsto \widetilde{P}^{-1} L\ \widetilde{P} = L_{\mathsf{comp1}} = \begin{pmatrix} \sigma_1 & 1 & \dots & 0 \\ \vdots & & \ddots & \\ \sigma_{n-1} & & & 1 \\ \sigma_n & 0 & \dots & 0 \end{pmatrix}.$$

Let us compare these two forms D and L_{comp1}. Which one is "better"?

Diagonalisation vs companion form

- Both D and L_{comp1} depend on n parameters

$$\lambda_1, \ldots, \lambda_n \quad \text{and} \quad \sigma_1, \ldots, \sigma_n$$

- These parameters are related in a very natural way: λ_i are the roots of the polynomial $t^n - \sigma_1 t^{n-1} - \cdots - \sigma_{n-1} t - \sigma_n$.
- For D, there are some issues with complex eigenvalues. No such problems with L_{comp1}, which is a real matrix.
- To find λ_i we first need to compute the characteristic polynomial of L, then find its roots (which is a very difficult task...). No such problem for L_{comp1}.
- Diagonalisation: to find the transition matrix P we need to find an eigenvector e_i for each eigenvalue λ_i by solving $(L - \lambda_i \cdot \mathrm{Id})\mathbf{x} = \mathbf{0}$.
- Companion form: to find the transition matrix $\widetilde{P}$ or, equivalently, the corresponding new basis vectors, we simply take $\tilde{e}_n = \xi$, $\tilde{e}_{n-1} = L\tilde{e}_n$, $\ldots$, $\tilde{e}_1 = L^{n-1}\tilde{e}_1$ with ξ being an (arbitrary) generic vector.
- Canonical basis in both cases is not unique, but depends on n parameters (rescaling of each eigenvector e_i and choosing ξ)
- Diagonal form D is much better for computations like D^n, $\exp D$, etc.

What do we want to know about gl-regular Nijenhuis operators?

Definition

A Nijenhuis operator $L(x)$ (on a manifold M) is called gl-*regular*, if it is gl-regular at each point $x \in M$.

Natural questions:

- Local canonical forms and local classification
- Perturbation of a Jordan block (in the class of Nijenhuis operators): which scenarios are allowed?
- Structure of the set of singular points
- Topological obstructions for existence of gl-regular Nijenhuis operators on compact manifolds
- Examples of regular Nijenhuis operators on compact manifolds

First companion form

Theorem

Let L be a real analytic gl-regular Nijenhuis operator with characteristic polynomial $\chi_L(\lambda) = \det(\lambda \cdot \mathrm{Id} - L) = \lambda^n - f_1\lambda^{n-1} - \dots - f_n$. Then in a suitable coordinate system $x = (x^1, \dots, x^n)$, this operator takes the form

$$L_{\mathrm{comp1}}(x) = \begin{pmatrix} f_1 & 1 & 0 & \dots & 0 \\ f_2 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ f_{n-1} & 0 & \dots & 0 & 1 \\ f_n & 0 & \dots & 0 & 0 \end{pmatrix}, \tag{1}$$

where $f_i = f_i(x)$ are the coefficients of the characteristic polynomial in this coordinate system. They satisfy the following PDE system:

$$\begin{aligned} \frac{\partial f_i}{\partial x^j} &= f_i \frac{\partial f_1}{\partial x^{j+1}} + \frac{\partial f_{i+1}}{\partial x^{j+1}}, \\ \frac{\partial f_n}{\partial x^j} &= f_n \frac{\partial f_1}{\partial x^{j+1}}. \end{aligned} \qquad i, j = 1, \dots, n-1. \tag{2}$$

Second companion form

Theorem

Let L be a real analytic gl-regular Nijenhuis operator with characteristic polynomial $\chi_L(\lambda) = \det(\lambda \cdot \mathrm{Id} - L) = \lambda^n - f_1\lambda^{n-1} - \cdots - f_n$. Then in a suitable coordinate system $x = (x^1, \dots, x^n)$, this operator takes the form

$$L_{\mathrm{comp2}}(x) = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \dots & 0 & 1 \\ f_n & f_{n-1} & \dots & f_2 & f_1 \end{pmatrix}, \tag{3}$$

where $f_i = f_i(x)$ are the coefficients of the characteristic polynomial in this coordinate system. They satisfy the PDE system that can be written in the form

$$\mathrm{d}\omega = 0, \quad \mathrm{d}\big(L^*\omega\big) = 0, \tag{4}$$

where $\omega = f_n \mathrm{d}x^1 + \cdots + f_1 \mathrm{d}x^n$.

Reducibility to a companion form is a strong condition

Question. Why not every gl-regular operator can be reduced to a companion form? In other words, does the Nijenhuis condition is really important?

Explanation. Companion form is parametrised by n functions (in n variables each). A coordinate change $y \mapsto x = x(y)$ is also parametrised by n functions (in n variables). However, an operator field L (which is not necessarily Nijenhuis) is parametrised by n^2 functions (in n variables).

Thus, a coordinate transformation can "eliminate" n functions, but $n^2 - n$ still survive. Hence, if $n^2 - n > n$ (i.e. starting from dimension 3) we can always find an operator that cannot be reduced to a companion form.

Example. The operator $L = \begin{pmatrix} 0 & 1 & 0 \\ -(y^2+1) & 0 & 1 \\ 0 & y^2+1 & 0 \end{pmatrix}$ defined in $\mathbb{R}^3(x, y, z)$ cannot be reduced to a companion form. Indeed, L is nilpotent (verify this!) and its companion form is the standard Jordan 0-block, i.e., a constant matrix. Hence, reducibility to companion form would imply that L is Nijenhuis, but it is not (verify this!).

Comments on (Nijenhuis) companion forms

- "Canonical" coordinate system $x = (x^1, \dots, x^n)$ is not unique. Moreover, the functions $f_1, \dots, f_n$ in the first column (last row) are not unique either.
- Formulas (1) and (3) should be viewed as pre-canonical forms. Local classification problem is open.
- Formulas (1) and (3) work both at generic and singular points. In particular $f_1, \dots, f_n$ may have a very complicated singularity.
- If L is differentially non-degenerate Nijenhuis operator (i.e., $f_1, \dots, f_n$ are independent), then there exist "distinguished" coordinate systems:
  - in the first formula $x^1 = f_1\ ,\ x^2 = f_2\ ,\ \dots\ ,\ x^n = f_n$,
  - in the second formula $x^1 = \operatorname{tr} L, x^2 = \frac{1}{2}\operatorname{tr} L^2, \dots, x^n = \frac{1}{n}\operatorname{tr} L^n$.
- (1) is equivalent to the existence of a vector field ξ such that $\xi, L\xi, \dots, L^{n-1}\xi$ are linearly independent and commute.
- (3) is equivalent to the existence of a function h such that the forms $\mathrm{d}h, L^*\mathrm{d}h, \dots, (L^*)^{n-1}\mathrm{d}h$ are linearly independent and closed.

Scalar invariants and differential relations between them

To each operator $L(x)$ on a manifold M we can naturally assign its scalar invariants. The most important of them are the coefficients f_i of the characteristic polynomial

$$\chi_L(\lambda) = \det(\lambda \cdot \mathrm{Id} - L) = \lambda^n - f_1(x)\lambda^{n-1} - f_2(x)\lambda^{n-2} - \cdots - f_{n-1}(x)\lambda - f_n(x),$$

and in particular $\det L = (-1)^n f^n$.

If L is Nijenhuis, their differentials $\mathrm{d}f_i$ satisfy some important relations.

Proposition

$$L^* \mathrm{d} \det L = \det L \cdot \mathrm{d} \operatorname{tr} L,$$

where $L^* : T_x^*M \to T_x^*M$ denotes the dual operator.

Proof.

Particular case: $L = \operatorname{diag}\big(u_1(x^1), \ldots, u_n(x^n)\big)$, then $\det L = \prod_i u_i(x^i)$, $\operatorname{tr} L = \sum_i u_i(x^i)$. Then the above identity takes the following form:

$$\begin{pmatrix} u_1 & & \\ & \ddots & \\ & & u_n \end{pmatrix} \begin{pmatrix} \frac{\partial u_1}{\partial x^1} & u_2 \ldots u_n \\ & \vdots & \\ u_1 \ldots u_{n-1} & & \frac{\partial u_n}{\partial x^n} \end{pmatrix} = u_1 \ldots u_n \begin{pmatrix} \frac{\partial u_1}{\partial x^1} \\ \vdots \\ \frac{\partial u_n}{\partial x^n} \end{pmatrix}.$$

Magic formula

$$\boxed{L^* \mathrm{d} \det L = \det L \cdot \mathrm{d} \operatorname{tr} L}$$

Replacing L with $\lambda \cdot \mathrm{Id} - L$ (which is also a Nijenhuis operator for each $t \in \mathbb{R}$) we get

$$\boxed{(\lambda \cdot \mathrm{Id} - L)^* \big(\mathrm{d}\chi(\lambda)\big) = -\chi(\lambda) \cdot \mathrm{d} f_1,}$$

where $\chi(\lambda) = \lambda^n - \sum_{i=1}^n f_i \lambda^{n-i}$. L.h.s. and r.h.s. of this identity are polynomials in λ. Equating the coefficients of these polynomials gives:

$$L^* \mathrm{d} f_i - \mathrm{d} f_{i+1} = f_i \, \mathrm{d} f_1 \quad (i = 1, \ldots, n-1), \qquad L^* \mathrm{d} f_n = f_n \, \mathrm{d} f_1.$$

Rewriting these n vector identities as one single matrix identity, we obtain

Proposition (Magic Formula)

Let L be a Nijenhuis operator. Then in any coordinate system $x^1, \ldots, x^n$, the following matrix identity holds:

$$\boxed{J \; L = L_{\mathrm{comp}1} J,}$$

where $J = \left(\frac{\partial f_i}{\partial x^j} \right)$ is the Jacobi matrix of the coefficients $f_1, \ldots, f_n$ of the characteristic polynomial of L.

Magic Formula and differentially non-degenerate operators

Magic Formula:

$$J\ L = L_{\mathrm{comp1}} J$$

Corollary

Assume that L is differentially non-degenerate, that is, $\det J \neq 0$, *then*

- *In any coordinate system, L can be reconstructed from the coefficients of its characteristic polynomial:*

$$L = J^{-1} L_{\mathrm{comp1}} J.$$

- *If we set $x^i = f_i$, i.e., take the coefficients of the characteristic polynomial as local coordinates, then L takes the form*

$$L = \begin{pmatrix} x^1 & 1 & & \\ \vdots & & \ddots & \\ x^{n-1} & & & 1 \\ x^n & 0 & \dots & 0 \end{pmatrix}.$$

Proof of the first companion form theorem

Exercise (Easy part)

Check that

$$L_{\text{comp1}}(x) = \begin{pmatrix} f_1 & 1 & 0 & \dots & 0 \\ f_2 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ f_{n-1} & 0 & \dots & 0 & 1 \\ f_n & 0 & \dots & 0 & 0 \end{pmatrix},$$

is Nijenhuis if and only if the following relations hold:

$$\boxed{\begin{aligned} \frac{\partial f_i}{\partial x^j} &= f_i \frac{\partial f_1}{\partial x^{j+1}} + \frac{\partial f_{i+1}}{\partial x^{j+1}}, \\ \frac{\partial f_n}{\partial x^j} &= f_n \frac{\partial f_1}{\partial x^{j+1}}. \end{aligned}}$$

or equivalently in matrix form

$$\begin{pmatrix} \frac{\partial f_1}{\partial x^j} \\ \vdots \\ \frac{\partial f_n}{\partial x^j} \end{pmatrix} = L_{\text{comp1}} \begin{pmatrix} \frac{\partial f_1}{\partial x^{j+1}} \\ \vdots \\ \frac{\partial f_n}{\partial x^{j+1}} \end{pmatrix}$$

Simply check the Nijenhuis condition

$$L^2[e_i, e_j] - L[Le_i, e_j] - L[e_i, Le_j] + [Le_i, Le_j] = 0$$

for any pair of basis vectors $e_i = \partial_{x^i}$, $e_j = \partial_{x^j}$ (obvious for $i, j > 1$).

Reducing to the first companion form

We have a gl-regular Nijenhuis operator $L(u)$ given in some coordinates $u^1, \dots, u^n$. Our goal is to find an invertible change of variables $x \leftrightarrow u$ that reduced L to companion form. This requirement is equivalent to

$$\left(\frac{\partial u}{\partial x}\right)^{-1} L \left(\frac{\partial u}{\partial x}\right) = L_{\mathrm{comp1}} \qquad \text{or} \qquad L \left(\frac{\partial u}{\partial x}\right) = \left(\frac{\partial u}{\partial x}\right) L_{\mathrm{comp1}}$$

Notice that if this relation holds for all the columns of the matrices in the l.h.s. and r.h.s. except for the first one, then for the first one this relation will be fulfilled automatically. This leads to

$$u_{x^{i-1}} = L\, u_{x^i} \quad \text{or, equivalently} \quad u_{x^{n-k}} = L^k\, u_{x^n},$$

a collection of quasilinear systems.

Questions to discuss: Can we resolve them? How many solutions are there? Is this system of PDEs compatible?

Quasilinear PDE systems of hydrodynamic type

By a system of PDEs of hydrodynamic type we understand $u_t = A(u)u_x$. In more detail:

$$\begin{pmatrix} u^1_t \\ u^2_t \\ \vdots \\ u^n_t \end{pmatrix} = \begin{pmatrix} a^1_1(u) & a^1_2(u) & \dots & a^1_n(u) \\ a^2_1(u) & a^2_2(u) & \dots & a^2_n(u) \\ \vdots & \vdots & \ddots & \vdots \\ a^n_1(u) & a^n_2(u) & \dots & a^n_n(u) \end{pmatrix} \begin{pmatrix} u^1_x \\ u^2_x \\ \vdots \\ u^n_x \end{pmatrix}, \tag{5}$$

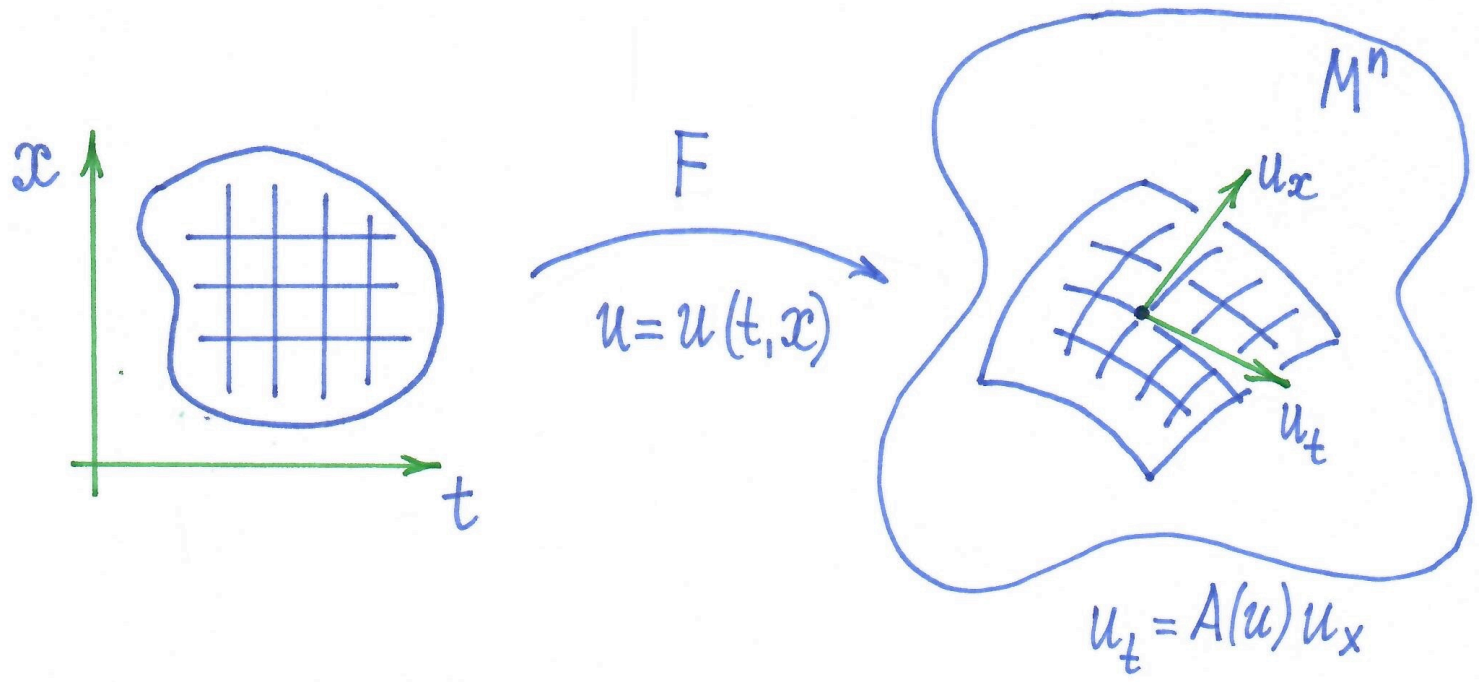
with $u^i(t,x)$ being unknown functions, $i = 1, \dots, n$.

Transformation $(u^1, \dots u^n) \mapsto (\tilde{u}^1, \dots, \tilde{u}^n)$ leads again to a quasilinear system of the form $\tilde{u}_t = \tilde{A}(\tilde{u})\tilde{u}_x$, where the matrices $\tilde{A}(\tilde{u})$ and $A(u)$ are related by means of the standard transformation rule for (1,1)-tensors:

$$\tilde{A}(\tilde{u}) = J\,A(u)\,J^{-1}, \quad J = \left(J^\alpha_\beta\right) = \left(\frac{\partial \tilde{u}^\alpha}{\partial u^\beta}\right).$$

Thus, $A(u)$ should be understood as a tensor field on a certain manifold M^n and $u^1, \dots, u^n$ as coordinates on this manifold. In this setting, equation (5) is geometric and invariant w.r.t. coordinate transforms.

Geometric interpretation



Solution of the hydrodynamic type system $u_t = A(u)u_x$ as a surface in the manifold M^n endowed with the field of endomorphisms A.

Important example and Conservation laws

Let M^n be a manifold and A a field of endomorphisms (operator) on it.

Consider $\gamma : S^1 \to M^n$ to be a closed loop in M^n. In coordinates, it is given parametrically as $\gamma(x) = \big(u^1(x), \dots, u^n(x)\big)$, $x \in \mathbb{R} \bmod 2\pi \simeq S^1$. Then the quasilinear equation

$$u_t = A\, u_x,$$

describes a certain evolution $\gamma_t : S^1 \to M^n$ of this loop in M determined by the operator A.

For a function $g : M \to \mathbb{R}$, we can consider a natural functional on the loop space defined by

$$\gamma \;\mapsto\; \oint_\gamma g\big(\gamma(x)\big)\,\mathrm{d}x$$

Definition

This functional (or the function g as its density) is called a conservation law for the PDE system $u_t = A\, u_x$ if for every solution $u(x,t)$ the value of this functional $\oint_\gamma g\big(u(t,x)\big)\,\mathrm{d}x$ does not change during evolution in t.

Conservation laws (continued...)

Proposition

A function $g : M \to \mathbb{R}$ defines a conservation law for $u_t = A\,u_x$ if and only if the differential 1-form $A^*(\mathrm{d}g)$ is exact, i.e., there exists a function f such that $A^*(\mathrm{d}g) = \mathrm{d}f$.

Here A^* denotes the dual operator $A^* : T_p^*M \to T_p^*M$, that is, $A^*(\mathrm{d}g) = A^i_j \frac{\partial g}{\partial u^i}\,\mathrm{d}u^i$.

Proof.

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^{2\pi} g(u(t,x))\,\mathrm{d}x = \int_0^{2\pi} \frac{\mathrm{d}}{\mathrm{d}t} g(u(t,x))\,\mathrm{d}x = \int_0^{2\pi} \frac{\partial g}{\partial u^i}\,u^i_t\,\mathrm{d}x$$
$$= \int_0^{2\pi} \frac{\partial g}{\partial u^i} A^i_j u^j_x\,\mathrm{d}x = \oint_\gamma \frac{\partial g}{\partial u^i} A^i_j\,\mathrm{d}u^j = \oint_\gamma A^*(\mathrm{d}g)$$

This integral vanishes on any loop if and only if $A^*(\mathrm{d}g)$ is exact.

Conservation laws for Nijenhuis operators

Let L be a Nijenhuis operator and consider the corresponding hydrodynamic type system

$$u_t = L(u)u_x$$

Proposition

1. The functions $f_k = \operatorname{tr} L^k$ are conservation laws for this PDE system. Moreover, $L^*(\mathrm{d}\operatorname{tr} L^k) = \frac{k}{k+1}\mathrm{d}\operatorname{tr} L^{k+1}$.
2. If M is simply connected, then every conservation law g_0 generate an infinite series of conservation laws g_1, g_2, ... such that

$$L^*(\mathrm{d}g_k) = \mathrm{d}g_{k+1}, \quad k = 0, 1, 2, \ldots$$

Proof

Nijenhuis identity $L^2[\xi,\eta] = L[L\xi,\eta] + L[\xi,L\eta] - [L\xi,L\eta] = 0$ can be rewritten as

$$\mathcal{L}_{L\xi} L = L\mathcal{L}_\xi L, \tag{6}$$

and

$$\mathrm{d}\big(L^{*2}\alpha\big)(\cdot,\cdot) = \mathrm{d}\big(L^*\alpha\big)(L\cdot,\cdot) + \mathrm{d}\big(L^*\alpha\big)(\cdot,L\cdot) - \mathrm{d}\alpha(L\cdot,L\cdot) \tag{7}$$

Now

$$(6) \overset{\text{by induction}}{\Rightarrow} \mathcal{L}_{L^k\xi} L = L^k \mathcal{L}_\xi L \;\Rightarrow\; \operatorname{tr}(\mathcal{L}_{L^k\xi} L) = \operatorname{tr}(L^k \mathcal{L}_\xi L) = \operatorname{tr}\left(\tfrac{1}{k+1}\mathcal{L}_\xi L^{k+1}\right)$$

$$\Rightarrow\quad \mathcal{L}_{L^k\xi}(\operatorname{tr} L) = \tfrac{1}{k+1}\mathcal{L}_\xi\big(\operatorname{tr} L^{k+1}\big) \quad\Rightarrow\quad (L^*)^k \mathrm{d}\operatorname{tr} L = \tfrac{1}{k+1}\mathrm{d}\operatorname{tr} L^{k+1},$$

which immediately implies the first statement.

Next, let $\alpha = \mathrm{d}g_0$ in (7), where g_0 is a conservation law so that $L^*\alpha = L^*\mathrm{d}g_0 = \mathrm{d}g_1$ and α are both exact and therefore closed. Then the r.h.s. of (7) vanishes and we get

$$\mathrm{d}(L^*\mathrm{d}g_1) = \mathrm{d}\big(L^{*2}\mathrm{d}g_0\big) = 0,$$

which implies existence of g_2 such that $L^*(\mathrm{d}g_1) = \mathrm{d}g_2$ and the second statement follows by induction.

Commuting flows (symmetry fields) for quasilinear PDEs

As explained, $u_t = A(u)u_x$ defines a flow in the space of (vector-) functions depending on x. Recall that we work with real-analytic objects.

For our quasilinear system $u_t = A(u)u_x$, we can apply Cauchy-Kovalevskaya theorem, which states that for any (local) initial condition $u(0, x)$ there exists a unique (local) solution $u(t, x)$.

Consider another flow $u_s = B(u)u_x$.
We will say that these flows commute if, speaking informally, the composition of the t-shift along A and s-shift along B does not depend on their order (for any initial condition!).

Equivalently, this means that for any initial condition $u(0, 0, x)$ there exists a solution $u(t, s, x)$ for the system of two equations $u_t = A(u)u_x$, $u_s = B(u)u_x$.

Proposition

The flows $u_t = A(u)u_x$ and $u_s = B(u)u_x$ commute if and only if

- $AB = BA$
- $\langle A, B\rangle(\xi, \xi) \stackrel{\text{def}}{=} [A\xi, B\xi] - B[A\xi, \xi] - B[\xi, A\xi] = 0$ for any vector field ξ.

Proof

We need to check that the following compatibility condition is fulfilled identically (in virtue of the system)

$$\frac{\partial}{\partial s}\left(A(u)u_x\right) - \frac{\partial}{\partial t}\left(B(u)u_x\right) = 0$$

We now compute the first term in the l.h.s. using $\frac{\partial}{\partial s}u^\alpha = B^\alpha_k u^k_x$, $\frac{\partial}{\partial s}u^k_j = \frac{\partial}{\partial x}u^j_s = \frac{\partial}{\partial x}\left(B^j_\alpha u^\alpha_x\right)$.

$$\frac{\partial}{\partial s}\left(A^i_j(u)u^j_x\right) = \frac{\partial A^i_j}{\partial u^\alpha}B^\alpha_k u^k_x u^j_x + A^i_j\frac{\partial}{\partial x}\left(B^j_\alpha u^\alpha_x\right) =$$

$$\frac{\partial A^i_j}{\partial u^\alpha}B^\alpha_k u^k_x u^j_x + A^i_j\left(\frac{\partial B^j_\alpha}{\partial u^\beta}u^\beta_x u^\alpha_x + B^j_\alpha u^\alpha_{xx}\right) = \left(\frac{\partial A^i_j}{\partial u^\alpha}B^\alpha_k + A^i_\alpha\frac{\partial B^\alpha_j}{\partial u^k}\right)u^k_x u^j_x + A^i_j B^j_\alpha u^\alpha_{xx}.$$

Similar computation for the second term $\frac{\partial}{\partial t}\left(B(u)u_x\right)$ leads to the relation

$$\left(\frac{\partial A^i_j}{\partial u^\alpha}B^\alpha_k + A^i_\alpha\frac{\partial B^\alpha_j}{\partial u^k} - \frac{\partial B^i_j}{\partial u^\alpha}A^\alpha_k - B^i_\alpha\frac{\partial A^\alpha_j}{\partial u^k}\right)u^k_x u^j_x + (A^i_j B^j_\alpha - B^i_j A^j_\alpha)u^\alpha_{xx} = 0.$$

Since this relation should be fulfilled identically for all initial conditions $u(0,0,x)$, the both terms vanish (for any u^k_x and u^α_{xx}) giving the two conditions from Proposition.

Hodograph method: finding solutions from a symmetry field

Hopf equation: $u_t = a(u)u_x$, where $u = u(t, x)$ is a scalar function.

How to solve? Take an arbitrary function $b(u)$ and consider the relation

$$x + t\, a(u) = b(u), \tag{8}$$

thinking of x, t and u as independent variables. Resolve this relation w.r.t. u to get $u = u(t, x)$. This function is a solution of Hopf equation.

Proof. Differentiating (8) w.r.t x and t gives:

$$\begin{aligned} 1 + t\, a'_u\, u_x &= b'_u\, u_x \\ a + t\, a'_u\, u_t &= b'_u\, u_t \end{aligned} \quad \text{or equivalently} \quad \begin{aligned} 1 &= (b'_u - t\, a'_u)\, u_x \\ a &= (b'_u - t\, a'_u)\, u_t \end{aligned}$$

implying $u_x = a\, u_t$, as required.

Proposition

Consider $u_t = A(u)u_x$ with $u = (u_1, \ldots, u_n)^\top$ and let $u_s = B(u)u_x$ be its symmetry field (commuting flow). Suppose $u(t, x)$ satisfies the matrix relation

$$x \cdot \mathrm{Id} + t \cdot A(u) = B(u).$$

Then $u(t, x)$ is a solution of $u_t = A(u)u_x$.

Commuting flows for Nijenhuis operators

Proposition

Let L be a Nijenhuis operator, then the flows

$$u_{t_k} = L^k(u)\, u_x, \quad k = 1, 2, 3, \dots$$

pairwise commute. In particular, the quasilinear system $u_t = L(u)u_x$ admits infinitely many commuting flows.

Corollary

Let L be a Nijenhuis operator. Then for any analytic matrix function $f(L)$ (e.g. polynomial with constant coefficients), the corresponding flow $u_s = f(L)u_x$ commutes with $u_t = L\, u_x$.

Proposition

Let L be a Nijenhuis operator, then the flows

$$u_{t_\lambda} = \frac{1}{\det(L - \lambda \cdot \mathrm{Id})} (L - \lambda \cdot \mathrm{Id})^{-1} u_x, \quad \lambda \in \mathbb{R}$$

pairwise commute.

The latter example leads to interesting examples of integrable PDEs in physics (e.g. dispersionless limit of multi-component KdV).

Integrability of quasilinear PDEs related to Nijenhuis operators

If L is diagonal with simple spectrum, then $u_t = L(u)u_x$ reduces to n uncoupled Hopf equations $u^k_t = \lambda(u^k)u^k_x$ and can be easily solved.

However, at those points where eigenvalues of L collide, this method does not work directly. In this case, it makes sense to look at the dynamics of the coefficients $f_1, \dots, f_n$ of the characteristic polynomial of L.

Proposition (follows immediately from Magic formula)

Let L be Nijenhuis. Then $u_t = L(u)u_x$ implies

$$\begin{pmatrix} \frac{\partial f_1}{\partial t} \\ \vdots \\ \frac{\partial f_n}{\partial t} \end{pmatrix} = L_{\text{comp1}}(f) \begin{pmatrix} \frac{\partial f_1}{\partial x} \\ \vdots \\ \frac{\partial f_n}{\partial x} \end{pmatrix}, \quad \text{where } L_{\text{comp1}}(f) = \begin{pmatrix} f_1 & 1 & & \\ \vdots & & \ddots & \\ f_{n-1} & & & 1 \\ f_n & 0 & \dots & 0 \end{pmatrix}.$$

Remark. In this setting, L is an arbitrary Nijenhuis operator, not necessarily gl-regular or differentially non-degenerate. Also L may have any kind of singularities. However, the dynamics of the coefficients of its characteristic polynomial is governed by a gl-regular and differentially non-degenerate operator $L_{\text{comp1}}(f)$!

Solving $f_t = L_{\text{comp1}}(f)\, f_x$ without integration

Our quasilinear system can be rewritten as $f_t = L_{\text{comp1}}(f)\, f_x$.

Fix arbitrary initial conditions $f_i(0, x) = v_i(x)$ and define the function

$$r(\lambda, \mu) = \lambda^n - v_1(\mu)\, \lambda^{n-1} - v_2(\mu)\, \lambda^{n-2} - \cdots - v_n(\mu) = 0.$$

Let $M = x \cdot \mathrm{Id} + t \cdot L$, where L is a gl-regular matrix and consider the matrix relation

$$r(M, L) = 0.$$

Then

- From this matrix relation, the coefficients $f_1, \ldots, f_n$ of the characteristic polynomial of L can be uniquely expressed in a neighbourhood of $(t, x) = (0, 0)$ as real analytic functions in t, x (by Implicit Function Theorem).
- The collection of functions $f(t, x) = \big(f_1(t, x), \ldots, f_n(t, x)\big)$ so obtained is the solution of $f_t = L_{\text{comp1}}(f)\, f_x$ with the given initial condition $f_i(0, x) = v_i(x)$.

Back to the proof of Companion Form Theorem

We just need to solve the system of quasilinear PDEs of the form

$$u_{x^{n-k}} = L^k u_{x^n}, \quad k = 1, \dots, n-1.$$

We know that these flows commute and, therefore, the solution exists for all initial conditions

$$u^i(0, \dots, 0, x^n) = h_i(x^n), \quad i = 1, \dots, n.$$

The initial condition can be chosen so that the Jacobi matrix $\left(\frac{\partial u}{\partial x}\right)$ is invertible. This completes the proof.
Also, similar to solving $f_t = L_{\text{comp1}}(f)\, f_x$ (see previous slide), we can solve the system of PDEs

$$f_{x^{n-k}} = L^k_{\text{comp1}}\, f_{x^n}, \quad k = 1, \dots, n,$$

that determines the functions in the first column of the companion Nijenhuis matrix. Hence, we get a "complete local description" of all gl-regular Nijenhuis operators.

Analytic functions of matrices (operators)

It can be easily checked that any polynomial $p(L)$, with constant coefficients, of a Nijenhuis operator L is a Nijenhuis operator too. Our goal is to generalise this property to the case of real analytic functions $f(L)$.

What kind of function we are going to consider? Examples include polynomials, matrix exponent $\exp A = \mathrm{Id} + A + \frac{1}{2}A + \frac{1}{3!}A^3 + \dots$, $\sin A$, $\sinh A$ and many others.

General construction. Suppose a compact set $K \subseteq \mathbb{C}$, a function $f : K \to \mathbb{C}$ and an operator L (on M) satisfy the following assumptions:

(i) $\mathbb{C} \setminus K$ is connected (we do not require that K is connected);

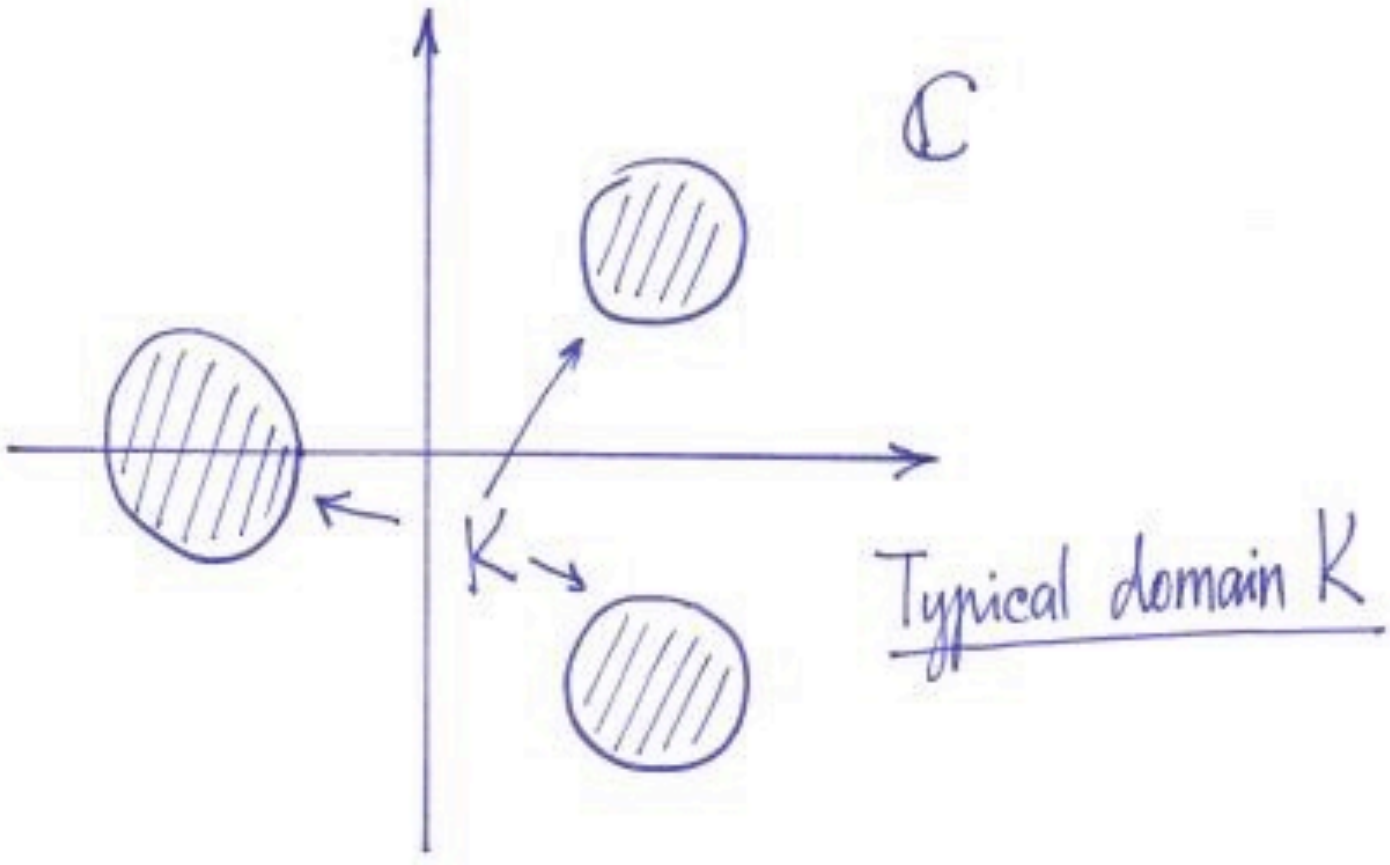
(ii) $f : K \to \mathbb{C}$ is a continuous function, and the restriction $f|_{\mathrm{int}(K)}$ is holomorphic;

(iii) K is symmetric with respect to the x-axes: for every $z \in K$ its conjugate $\bar{z}$ also belongs to K.

(iv) $f(z) = \bar{f}(\bar{z})$ for every $z \in K$;

(v) for every $x \in M$ we have $\mathrm{Spectrum}_x L \subset \mathrm{int}(K)$.

Figure



Analytic functions of matrices

Under the assumptions (i–v), one can naturally define an operator $f(L)$[1].

Mergelyan's theorem. The function f can be uniformly approximated by polynomials with real coefficients.

We define $f(L) = \lim_{k\to\infty} p_k(L)$.

One can show that the limit exists, is independent on the choice of the sequence p_k, smoothly depends on $x \in M^n$ (actually, the function $f(L)$ is analytic in the entries of L), and behaves as a $(1,1)$-tensor under coordinate transformations. Moreover, $\mathrm{Spectrum}_x f(L) = f(\mathrm{Spectrum}_x L)$.

Finally, since uniform convergence of holomorphic functions implies convergence of derivatives, we also have $\mathcal{L}_\xi\big(f(L)\big) = \lim_{k\to\infty} \mathcal{L}_\xi\big(p_k(L)\big)$ where $\mathcal{L}_\xi$ denotes the Lie derivative (in particular, a similar relation holds for partial derivatives).

Theorem

Let L be a Nienhuis operator. Then $f(L)$ is a Nijenhuis operator for any function $f : K \to \mathbb{C}$ satisfying assumptions (i)–(v).

---

[1] N. J. Higham, *Functions of matrices. Theory and computation*, SIAM, Philadelphia, PA, 2008.

Application: canonical complex structure

Consider the following function on $\mathbb{C} \setminus \mathbb{R}$:

$$f(z) = \begin{cases} \mathrm{i}\,, & \text{if } \operatorname{Im} z > 0, \\ -\mathrm{i}\,, & \text{if } \operatorname{Im} z < 0. \end{cases}$$

This function is locally constant and holomorphic on $\mathbb{C} \setminus \mathbb{R}$. It is easy to see that all properties (i)–(iv) are fulfilled for any symmetric compact subset $K \subset \mathbb{C} \setminus \mathbb{R}$.

Proposition

Let L be an operator on a real manifold M with no real eigenvalues. Then the operator

$$J = f(L) \tag{9}$$

is well defined on M and satisfies the following properties:

1. J is smooth;
2. $J^2 = -\mathrm{Id}$, i.e. J is an almost complex structure on M;
3. $JL = LJ$, i.e. L is a complex linear operator w.r.t. J;
4. if L is Nijenhuis, then J is Nijenhuis too and hence is a complex structure on M.

Generalised Nirenberg-Newlander theorem

Theorem

Let L be a Nijenhuis operator on M with no real eigenvalues, i.e., its spectrum at every point $x \in M$ belongs to $\mathbb{C} \setminus \mathbb{R}$. Then

1. *M is a complex manifold w.r.t. a complex structure J canonically associated with L.*
2. *L is a complex holomorphic tensor field on M w.r.t. J, i.e. can be written in the form*
$$L^{\mathbb{C}} = \sum_{i,j=1}^{n} l^i_j(z)\, \mathrm{d}z^j \otimes \partial_{z^i}$$
with all the functions $l^i_j(z)$ being holomorphic in complex coordinates $z_1, \ldots, z_n$.
3. *The complex Nijenhuis torsion of L vanishes, i.e.*
$$\left(\mathcal{N}^{\mathbb{C}}_L\right)^i_{jk} = l^m_j \frac{\partial l^i_k}{\partial z^m} - l^m_k \frac{\partial l^i_j}{\partial z^m} - l^i_m \frac{\partial l^m_k}{\partial z^j} + l^i_m \frac{\partial l^m_j}{\partial z^k} = 0.$$

Useful exercise

Consider the operator

$$L = \begin{pmatrix} u(x,y) & -v(x,y) \\ v(x,y) & u(x,y) \end{pmatrix}.$$

Notice that in terms of the complex coordinate $z = x + iy$, this operator can be written as 1×1 complex matrix $\Big(f(z)\Big)$, where
$f(z) = u(z) + \mathrm{i}\, v(z)$

Check that this operator is Nijenhuis if and only if $f(z)$ is a holomorphic function.

Some global results

Theorem

Let L be a Nijenhuis operator on a closed connected manifold M with a non-real eigenvalue $\lambda \in \mathbb{C} \setminus \mathbb{R}$ at least at one point. Then this number λ is an eigenvalue of L with the same algebraic multiplicity at every point of M.

Shortly: a Nijenhuis operator on a closed manifold cannot have non-constant complex eigenvalues.

Corollary

A Nijenhuis operator L on a closed manifold cannot have differentially non-degenerate singular points (like e.g. the point $(x_1, x_2) = (0, 0)$ for $L = \begin{pmatrix} x_1 & 1 \\ x_2 & 0 \end{pmatrix}$).

Corollary

The eigenvalues of a Nijenhuis operator on the 4-dimensional sphere S^4 are all real.