Outline of Lecture 4

- Compatible geometric (and algebraic) structures
- Nijenhuis operators as partners of other geometric structures
- Nijenhuis pencils
- Compatible Poisson structures and bi-Hamiltonian systems
- $\infty$-dimensional Poisson brackets
- Non-homogeneous brackets of type $B_h + A_g$
- Relation to Frobenius pencils
- Main example: AFF-pencil
- Integrable PDE systems related to AFF-pencil
- Classification theorem for compatible Poisson brackets of type $B_h + A_g$
Compatible structures

In mathematics, very often when working with a certain class $\mathcal{A}$ of objects, we want to distinguish those pairs $A, \tilde{A} \in \mathcal{A}$ that are compatible in some sense. (This relation is reflexive and symmetric, but not necessarily transitive!)

Examples include:

1. compatible Poisson structures
2. geodesically equivalent metrics
3. flat metrics (in the context of Poisson brackets of hydrodynamic type)
4. compatible algebras defined by a certain property (Lie, associative, Frobenius)
5. compatible Nijenhuis operators

It is interesting that in cases 1, 4 and 5 the definition of compatibility is the same: $A \in \mathcal{A}$ and $\tilde{A} \in \mathcal{A}$ are compatible if $A + \tilde{A} \in \mathcal{A}$. For instance, two associative algebras $(a, \ast)$ and $(a, \ast)$ defined on the same vector space $a$ are compatible, if the new multiplication $\ast + \ast$ defined by

$$\xi, \eta \in a \quad \mapsto \quad \xi \ast \eta + \xi \ast \eta$$

is still associative.
Nijenhuis operators as partners of other geometric structures

- Consider two **compatible** Poisson structures $P_1$ and $P_2$, one of which is non-degenerate. Then the operator $L = P_2 P_1^{-1}$ is Nijenhuis.

- Consider two **geodesically equivalent** metrics $g_1$ and $g_2$. Then the operator $L = \left( \frac{\det g_2}{\det g_1} \right)^{\frac{1}{n+1}} g_2^{-1} g_1$ is Nijenhuis.

- Consider two **compatible** Poisson brackets of **hydrodynamic type**, determined by two flat (contravariant) metrics $g_1$ and $g_2$. Then the operator $L = g_1 g_2^{-1}$ is Nijenhuis.

For each of these situations it makes sense to introduce the following terminology:

- $L$ is **compatible** with a Poisson structure $P$, if $P$ and $LP$ are compatible Poisson structures.

- $L$ is **geodesically compatible** with $g$, if the metrics $g$ and $\frac{1}{|\det L|} gL^{-1}$ are geodesically equivalent.

- $L$ is **Poisson compatible** with a flat (contravariant) metric $g$, if $Lg$ is flat and the metrics $g$ and $Lg$ define a pair of compatible Poisson brackets of hydrodynamic type.
Which Nijenhuis $L$ can find a suitable partner?

**Problem**

Let $L$ be a Nijenhuis operator. Does it admit

- compatible Poisson structure $P$,
- geodesically compatible metric $g$,
- Poisson compatible (flat) metric $g$?

This question makes sense both in global and local context.

**Comments.**

- If the algebraic type of an operator $L$ is locally constant (i.e. we consider a non-singular point), then the answer is essentially known in the first (Turiel) and second (V. Matveev, AB) cases.
- Thus the question is basically related to singular points: which singularities of the operator $L$ are admissible in this context?
- If $L$ is differentially non-degenerate at a given singular point, then the partner for $L$ exists in each of the above cases.
- In the case of geodesic compatibility, all admissible singularities are known if $g$ is positive definite (V. Matveev).
In each of the above mentioned cases:

- compatible Poisson brackets
- compatible flat metrics
- geodesically equivalent metrics

it makes sense to consider not just two, but a multi-parametric family $\mathcal{P}$ of pairwise compatible structures.

Since each pair of compatible objects is related by means of a Nijenhuis operator, together with $\mathcal{A}$ we obtain a natural family $\mathcal{L}$ of Nijenhuis operators:

$$\mathcal{P} = \{P_c\} \quad \mapsto \quad \mathcal{L} = \{L_c\}$$

Namely, we first choose a certain (non-degenerate) “reference point” $P \in \mathcal{P}$ and then take all $L_c$’s that link $P$ with all the other elements $P_c \in \mathcal{P}$.

**Important observation:** $\mathcal{L} = \{L_c\}$ is a vector space that consists of Nijenhuis operators (= Nijenhuis pencil).

**Definition**

Two Nijenhuis operators $L_1$ and $L_2$ are called **compatible** if their sum $L_1 + L_2$ is a Nijenhuis operator also.
Compatible Nijenhuis operators

Consider a pencil of compatible structures $\mathcal{P} = \{P_c\}_{c \in \mathbb{R}^m}$ (e.g. Poisson structures). As discussed, we can consider a natural Nijenhuis pencil associated to it:

$$\mathcal{L} = \{L_c\}_{c \in \mathbb{R}^n}, \quad \text{where} \quad L_c = P_cP^{-1}$$

for some fixed $P \in \mathcal{P}$ so that $\mathcal{P} = \mathcal{L}P$. Notice that the choice of the “reference point” $P \in \mathcal{P}$ is not unique.

Take three compatible structures $P_0$, $P_1$ and $P_2$. Then $L_1 = P_1P_0^{-1}$ and $L_2 = P_2P_0^{-1}$ are compatible Nijenhuis operators. On the other hand, since $P_1$ and $P_2$ are compatible also, then the operator $P_1P_2^{-1} = L_1L_2^{-1}$ is Nijenhuis. This “observation” suggests that compatibility of two Nijenhuis operators $L_1$ and $L_2$ is somehow related to the fact that $L_1L_2^{-1}$ is Nijenhuis. Indeed, we have

**Lemma**

*Consider a pair of non-degenerate Nijenhuis operators $L_1$ and $L_2$. Then $L_1$ and $L_2$ are compatible if and only if $L_1L_2^{-1}$ is Nijenhuis operator.*
Proof

We need to analyse the Nijenhuis condition $N_{L_1+L_2} = 0$. It is easy to see that $N_{L_1+L_2} = N_{L_1} + N_{L_2} + N_{L_1,L_2}$, where the last term (known as Frolicher-Nijenhuis bracket of $L_1$ and $L_2$) takes the form:

$$N_{L_1,L_2}(v, w) = L_1 L_2[v, w] + L_2 L_1[v, w] + [L_1 v, L_2 w] + [L_2 v, L_1 w] - L_1 [L_2 v, w] - L_1 [v, L_2 w] - L_2 [L_1 v, w] - L_2 [v, L_1 w]$$

Let us prove the following identity, which immediately implies the statement of the lemma:

$$N_{L_1}(v, w) + L_1 L_2^{-1} L_1 L_2^{-1} N_{L_2}(v, w) - N_{L_1 L_2^{-1}}(L_2 v, L_2 w) = L_1 L_2^{-1} N_{L_1, L_2}(v, w).$$

We have

$$N_{L_1}(v, w) + L_1 L_2^{-1} L_1 L_2^{-1} N_{L_2}(v, w) - N_{L_1 L_2^{-1}}(L_2 v, L_2 w) =$$

$$= L_1 L_2^{-1} L_2 L_1[v, w] + [L_1 v, L_1 w] - L_1 L_2^{-1} L_2 [L_1 v, w] - L_1 L_2^{-1} L_2 [v, L_1 w] +$$

$$+ L_1 L_2^{-1} L_1 L_2[v, w] + L_1 L_2^{-1} L_1 L_2^{-1} [L_2 v, L_2 w] - L_1 L_2^{-1} L_1 [L_2 v, w] - L_1 L_2^{-1} L_1 [v, L_2 w]$$

$$- L_1 L_2^{-1} L_1 L_2^{-1} [L_2 v, L_2 w] - [L_1 v, L_1 w] + L_1 L_2^{-1} [L_1 v, L_2 w] + L_1 L_2^{-1} [L_2 v, L_1 w] =$$

$$= L_1 L_2^{-1} N_{L_1, L_2}(v, w),$$

as stated.
Definition. A pencil of Nijenhuis operators is a vector subspace $\mathcal{L} \subset \text{End}(TM)$, which consist of Nijenhuis operators.

Examples:
- all constant operators $\{A = (a^i_j), \ a^i_j \in \mathbb{R}\}$
- all scalar operators $\{A = f(x) \cdot \text{Id}, \ f \in C^\infty(M)\}$

A natural general question is description of examples and classification of Nijenhuis pencils (at generic points).

Maximal pencils are of particular interest (i.e. those which cannot be extended).

Examples:
- diagonal pencil $\{L = \text{diag}(f_1(x_1), f_2(x_2), \ldots, f_n(x_n))\}$
- pencil of operators of the form $\{L = \left(a_{ij} + b_i x_j + x_i b_j + a x_i x_j\right), a_{ij} = a_{ji}\}$

Since we do not know almost anything about Nijenhuis pencils, a natural task, at the moment, would be just finding and studying examples.
Poisson structures and Hamiltonian systems (see Lecture 3)

Definition
A **Poisson structure** is defined to be a skew-symmetric tensor $P^{ij}$ of type $(2, 0)$ such that the corresponding operation (**Poisson bracket**) \[ \{\cdot, \cdot\} : \mathcal{C}^\infty(M) \times \mathcal{C}^\infty(M) \rightarrow \mathcal{C}^\infty(M), \]
\[ f, g \mapsto \{f, g\} = P(d f, d g) = \sum P^{ij} \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j}, \]
satisfies the Jacobi identity:
\[ \{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0 \text{ for all } f, g, h \in \mathcal{C}^\infty(M). \]

Remark. If $\det P \neq 0$, then we can define the inverse tensor $\omega = P^{-1}$ which is a differential 2-form. The Jacobi identity, in this case, means exactly that $d \omega = 0$, i.e., $\omega$ is a symplectic structure.

Definition
Let $H$ be a smooth function. The vector field
\[ \mathcal{X}_H = P(d H, \cdot), \quad \text{(equivalently } \mathcal{X}_H^j = P^{ij} \frac{\partial H}{\partial x^i} \text{ or } \mathcal{X}_H = P \, d H) \]
is said to be **Hamiltonian** (w.r.t. $P$ and with the **Hamiltonian function** $H$).
Definition
Two Poisson structures $P$ and $\tilde{P}$ are called \textit{compatible}, if their sum $P + \tilde{P}$ is also a Poisson structure.

Definition
A vector field $\mathcal{X}$ is called \textit{bi-Hamiltonian}, if it is Hamiltonian with respect to two compatible Poisson brackets $P$ and $\tilde{P}$, that is,

$$\mathcal{X} = P \, d \, H = \tilde{P} \, d \, \tilde{H}.$$  

Question. Why are we interested in bi-Hamiltonian systems? What are their properties? And why are such systems “better” than the others?

We will assume here that at least one of the compatible Poisson structures $P$ and $\tilde{P}$ is non-degenerate so that e.g. $P^{-1}$ makes sense.
Basic properties

Consider the linear operator $L = \tilde{P}P^{-1}$ or, equivalently,

$$P(L^* df, dg) = \tilde{P}(df, dg) \quad \text{for all } f, g \in C^\infty(M).$$

**Proposition**

Let $P$ and $\tilde{P}$ be Poisson structures. Then $P$ and $\tilde{P}$ are compatible if and only if $L = \tilde{P}P^{-1}$ is a Nijenhuis operator.

In the theory of bi-Hamiltonian systems, the operator $L$ plays an important role and is known as *recursion operator*.

In particular, notice the following formula for Hamiltonian vector fields:

$$\tilde{X}_f = LX_f.$$

Also notice that without loss of generality we may, at least locally, assume that $L$ is non-degenerate. Indeed, replacing $\tilde{P}$ with $\tilde{P} + \lambda P$ leads to $L \mapsto L + \lambda \text{Id}$. 
We first notice that the Jacobi identity for a Poisson structure $\mathcal{P}$ can be rewritten, in terms of Hamiltonian vector fields, as follows:

$$\mathcal{X}_{\{f,g\}} = [\mathcal{X}_f, \mathcal{X}_g].$$

Hence, the Jacobi identity for $\mathcal{P} + \mathcal{\tilde{P}}$ can be rewritten as follows (below we use the fact that $\mathcal{P}$ and $\mathcal{\tilde{P}}$ are Poisson and remove the terms that vanish automatically due to this fact):

$$\mathcal{\tilde{X}}_{\{f,g\}} + \mathcal{X}_{\{f,g\}} = [\mathcal{\tilde{X}}_f, \mathcal{X}_g] + [\mathcal{X}_f, \mathcal{\tilde{X}}_g]$$

Now we use the fact that $\mathcal{\tilde{X}}_f = L\mathcal{X}_f$ for any $f$:

$$L\mathcal{X}_{\{f,g\}} + L^{-1}\mathcal{\tilde{X}}_{\{f,g\}} = [L\mathcal{X}_f, \mathcal{X}_g] + [\mathcal{X}_f, L\mathcal{X}_g]$$

Finally we use again the fact that $\mathcal{P}$ and $\mathcal{\tilde{P}}$ are both Poisson to get

$$L[\mathcal{X}_f, \mathcal{X}_g] + L^{-1}[L\mathcal{X}_f, L\mathcal{X}_g] = [L\mathcal{X}_f, \mathcal{X}_g] + [\mathcal{X}_f, L\mathcal{X}_g]$$

Multiplying this identity by $L$ we get the Nijenhuis relation for $L$. It remains to recall that $\mathcal{P}$ is non-degenerate so that Hamiltonian vector fields generate the whole tangent space.
Proposition

Let $\mathcal{X}$ be a bi-Hamiltonian vector field w.r.t. $P$ and $\mathcal{P}$. Then the coefficients $\sigma_k$ of the characteristic polynomial of the recursion operator $L = \mathcal{P}P^{-1}$ are first integrals of $\mathcal{X}$. Moreover, these integrals commute with respect to the both structures $P$ and $\mathcal{P}$.

Proof. 1. First integrals. It is a well known fact that $P$ is preserved by each Hamiltonian vector field $\mathcal{X}_f$, i.e., $\mathcal{L}_{\mathcal{X}_f}P = 0$. Since in our case $\mathcal{X}$ is bi-Hamiltonian, we conclude that $\mathcal{X}$ preserves both $P$ and $\mathcal{P}$. Hence, $\mathcal{X}$ also preserves the recursion operator $L = \mathcal{P}P^{-1}$, and therefore all of its algebraic invariants such as coefficients $\sigma_k$ of $\chi_L(t)$.

2. Commutativity. Instead of $\sigma_k$, we check commutativity of $f_k = \frac{1}{k} \text{tr} L^k$, i.e., the identities $\{f_k, f_m\} = 0$. We use the following property of Nijenhuis operators $d f_k = L^* d f_{k-1}$ which implies:

$$\{f_k, f_m\} = P(d f_k, d f_m) = P(L^* d f_{k-1}, d f_m) = \{f_{k-1}, f_m\} = \{f_{k-1}, f_{m+1}\}$$

and by induction $\{f_k, f_m\} = \{f_{k-s}, f_{m+s-1}\} = \{f_{k-s}, f_{m+s}\}$. It remains to notice that for a suitable $s$ we have either $k-s = m+s$ or $k-s = m+s-1$ and the statement follows from skew symmetry of $P$ and $\mathcal{P}$. 
Applications to $\infty$-dim Poisson Structures

Poisson structure, Hamiltonian $\mapsto$ Hamiltonian system of ODEs

$$H(x), P^{ij}(x) \mapsto \frac{dx^i}{dt} = P^{ij} \frac{\partial H}{\partial x^j}$$

$\infty$-dimensional (differential-geometric) Poisson brackets, Hamiltonian functional (differential polynomial) in $u^1, \ldots, u^n \mapsto$ Hamiltonian system of PDEs

$$H(u, u_x, \ldots), \mathcal{P}^{ij} \mapsto \frac{\partial u^i}{\partial t} = \mathcal{P}^{ij} \left[ \frac{\delta H}{\delta u^j} \right] = F^i(x, u, u_x, u_{xx}, \ldots)$$

**Darboux theorem:** Poisson bracket can be reduces to constant form $P^{ij} = \text{const}$

$\infty$-dimensional case: ‘sometimes’ Poisson bracket can be reduced to a constant form (so called Poisson-Darboux structures)

**Examples:**

$$\mathcal{P}^{ij} = g^{ij} D \quad \mapsto \quad \mathcal{P} = A_g = g^{ij}(u) D - \Gamma^j_s u^s_x$$

$$\mathcal{P}^{ij} = h^{ij} D^3 \quad \mapsto \quad \mathcal{P} = B_h = h^{ij}(u) D^3 - 3 \Gamma^j_s u^s_x D^2 + \ldots$$
Statement of the problem

Recall that two Poisson brackets $\mathcal{P}_1$ and $\mathcal{P}_2$ are \textit{compatible}, if the sum $\mathcal{P}_1 + \mathcal{P}_2$ is a Poisson bracket also.

\begin{itemize}
  \item \textbf{Problem 1.} Describe pairs of flat metrics $h, g$ such that $\mathcal{B}_h + \mathcal{A}_g$ is a Poisson bracket (equivalently, $\mathcal{B}_h$ and $\mathcal{A}_g$ are compatible).
  
  \item \textbf{Problem 2.} Describe compatible Poisson brackets of the form $\mathcal{B}_h + \mathcal{A}_g$ and $\mathcal{B}_{\bar{h}} + \mathcal{A}_{\bar{g}}$. Equivalently, describe quadruples of flat metrics $g, \bar{g}, h, \bar{h}$ such that ...
  
  \item \textbf{Problem 2’.} Describe pairs of flat metrics $g$ and $\bar{g}$, for which we can find another pair of flat metrics $h$ and $\bar{h}$ such that $\mathcal{B}_h + \mathcal{A}_g$ and $\mathcal{B}_{\bar{h}} + \mathcal{A}_{\bar{g}}$ are Poisson and compatible. This “procedure” $A_g, A_{\bar{g}} \mapsto \mathcal{B}_h + \mathcal{A}_g, \mathcal{B}_{\bar{h}} + \mathcal{A}_{\bar{g}}$ is known as dispersive perturbation.
\end{itemize}

\textbf{Motivation.} Integrability of many multicomponent PDE systems is related to bi-Hamiltonian structures of this kind. Examples include multicomponent KdV, Camassa-Holm and Harry Dym equations.
Definition (B. Dubrovin)

Two contravariant flat metrics $g$ and $\bar{g}$ are Poisson compatible if:

1. $\hat{g}^{ij} = g^{ij} + \bar{g}^{ij}$ is flat,
2. $\hat{\Gamma}^{ij}_s = \Gamma^{ij}_s + \bar{\Gamma}^{ij}_s$.

This condition is equivalent to compatibility of the corresponding first order Poisson brackets $A_g$ and $A_{\bar{g}}$. One also says that $\{\lambda g^{ij} + \mu \bar{g}^{ij}\}$ is a flat two-dimensional pencil of (contravariant) metrics. In a similar way, we can define flat pencils $\mathcal{G} = \{g^{ij}_c\}_{c \in \mathbb{R}^n}$ of any dimension $m$.

Fact (E. Ferapontov)

If $g$ and $\bar{g}$ are Poisson compatible, then $R = \bar{g}g^{-1}$ is a Nijenhuis operator.

Hence, every flat pencil can be written as $\mathcal{G} = \{R_c \ g\}_{c \in \mathbb{R}^m}$, where $\mathcal{R} = \{R_c\}_{c \in \mathbb{R}^m}$ is a Nijenhuis pencil.
Frobenius pencils

Definition
A commutative associative algebra \( (\mathfrak{a}, \ast) \) is called Frobenius if it is endowed with a nondegenerate symmetric bilinear form \( b(\ , \ , ) \) such that

\[
b(\xi \ast \eta, \zeta) = b(\xi, \eta \ast \zeta), \quad \text{for all } \xi, \eta, \zeta \in \mathfrak{a}.
\]

Remarkable property (A. Balinskii, S. Novikov)
Let \( e^1, \ldots, e^n \) be a basis of \( \mathfrak{a} \) and \( e^i \ast e^j = \sum a^i_j e^s, \ b(e^i, e^j) = b^i_j \). Then the contravariant metric

\[
g^{ij}(u) = b^{ij} + a^{ij}_s u^s \tag{1}
\]

is flat.

Definition
A flat pencil \( \mathcal{G} = \{g_c\}_{c \in \mathbb{R}^n} \) is called Frobenius if there exists a coordinate system \( u^1, \ldots, u^n \) such that every metric \( g_c \in \mathcal{G} \) takes the form (1) for a certain Frobenius algebra \( (\mathfrak{a}_c, b_c), \ c \in \mathbb{R}^n \).
Theorem

Let $g$ and $h$ be two flat metrics. Then $\mathcal{B}_h + \mathcal{A}_g$ is Poisson if and only if there exists a coordinate system $u^1, \ldots, u^n$ such that:

1. $g^{\alpha\beta}(u) = b^{\alpha\beta} + a^{\alpha\beta}_s u^s$, and $u^1, \ldots, u^n$ are Frobenius for $g$;
2. $h^{\alpha\beta} = \text{const}$;
3. $h = (h^{\alpha\beta})$ is a Frobenius form for $\alpha$.

The coordinates $(u^1, \ldots, u^n)$ are called Frobenius for $\mathcal{B}_h + \mathcal{A}_g$.

Frobenius coordinates are not unique: they are defined up to affine transformations.
Pre-solution of Problem 2 in terms of Frobenius pencils

Theorem
Consider two non-homogeneous Poisson structures $\mathcal{B}_h + \mathcal{A}_g$ and $\mathcal{B}_\bar{h} + \mathcal{A}_\bar{g}$ and suppose that $R = \bar{h} h^{-1}$ has $n$ different eigenvalues.

They are compatible if and only if

(i) the pencil $\{\lambda g^{ij} + \mu \bar{g}^{ij}\}$ is Frobenius, i.e., take form (1) in suitable coordinates $u^1, \ldots, u^n$;

(ii) in these coordinates, $\lambda \bar{h}^{ij} + \mu h^{ij}$ are constant and define another Frobenius form for the Frobenius algebra related to $\lambda g^{ij} + \mu \bar{g}^{ij}$.

Moreover, if $\{\lambda g^{ij} + \mu \bar{g}^{ij}\}$ is a Frobenius pencil, i.e., (i) holds, then we can always find $h$ and $\bar{h}$ satisfying (ii).

Conclusion. The problem reduces to classification of two-dimensional Frobenius pencils.
Consider a real affine space \( V \cong \mathbb{R}^n \) with coordinates \( u^1, \ldots, u^n \) and define Nijenhuis operator \( L \) and contravariant metric \( g_0 \) on it by:

\[
L = \begin{pmatrix}
  u^1 & 1 & 0 & \cdots & 0 \\
  u^2 & 0 & 1 & \cdots & 0 \\
  & \ddots & \ddots & \ddots & \ddots \\
  u^{n-1} & 0 & 0 & \cdots & 1 \\
  u^n & 0 & 0 & \cdots & 0
\end{pmatrix}, \quad g_0 = \begin{pmatrix}
  0 & 0 & \cdots & 0 & 0 & 1 \\
  0 & 0 & \cdots & 0 & 1 & -u^1 \\
  0 & 0 & \cdots & 1 & -u^1 & -u^2 \\
  & \ddots & \ddots & \ddots & \ddots & \ddots \\
  0 & 1 & \cdots & -u^{n-4} & -u^{n-3} & -u^{n-2} \\
  1 & -u^1 & \cdots & -u^{n-3} & -u^{n-2} & -u^{n-1}
\end{pmatrix}.
\]

Consider

\[
\{ g = P(L)g_0 \}, \quad \text{where } P(\cdot) \text{ is a polynomial of degree } \leq n
\]

This is a Frobenius pencil, i.e., \( (u^1, \ldots, u^n) \) are common Frobenius coordinates for all \( g \)'s.

It is a remarkable fact that for each \( g = P(L)g_0 \) we can find a partner \( h \) such that \( B_h + A_g \) is a Poisson structure and all of them are compatible (Antonowicz, Fordy):

\[
\begin{align*}
g^{\alpha\beta} &= b^{\alpha\beta} + a^{\alpha\beta}_s u^s \\
&\quad \rightarrow \quad h^{\alpha\beta} = h^{\alpha\beta}(g, m) = m^0 \cdot b^{\alpha\beta} + a^{\alpha\beta}_s m^s
\end{align*}
\]

where \( m^0, m^1, \ldots, m^n \) are some constants.
As a result we obtain an \( n + 1 \) dimensional pencil of non-homogeneous Poisson structures of the form

\[
\mathcal{F}_m = \{ \mathcal{B}_h + \mathcal{A}_g \mid g = P(L)g_0, h = h(g, m) \}
\]

Here \( P(\cdot) \) serves as a parameter of the bracket within the AFF-pencil, whereas \( m \) parametrises dispersive perturbations of this pencil.

AFF pencil in diagonal coordinates (Ferapontov–Pavlov): Consider the eigenvalues of \( L \) as local coordinates \( x^1, \ldots, x^n \). In these coordinates, \( g_0 \) and \( L \) become both diagonal:

\[
g_0 = g_{\text{LC}} = \sum_{i=1}^{n} \left( \prod_{s \neq i} (x^i - x^s) \right)^{-1} \left( \frac{\partial}{\partial x^i} \right)^2, \quad L = \text{diag}(x^1, \ldots, x^n), \quad (2)
\]

so that the AFF pencil becomes diagonal too:

\[
\{ P(L)g_{\text{LC}} \}, \quad \text{where } P(\cdot) \text{ is a polynomial of degree } \leq n. \quad (3)
\]

Transition from diagonal to Frobenius coordinates is natural: \( u_i \) are elementary symmetric polynomials in \( x^1, \ldots, x^n \).
Classification in the ‘generic’ case

Our goal is to classify pairs of compatible flat metrics $g$ and $\bar{g}$ that admit a common Frobenius coordinate system (equivalently, Frobenius pencils).

**Theorem**

Let $\{\lambda g + \mu \bar{g}\}$ be a Frobenius pencil. Assume that the eigenvalues of the Nijenhuis operator $R = \bar{g}g^{-1}$ are all different and in the diagonal coordinates (such that $R$ is diagonal) every diagonal component of $g$ depends on all variables. Then this pencil is contained in the AFF-pencil, in other words, there exists a coordinate system $(x_1, \ldots, x^n)$ such that

$$g = P(L)g_0 \quad \text{and} \quad \bar{g} = Q(L)g_0.$$  

for some polynomials $P(\cdot)$ and $Q(\cdot)$ of degree $\leq n$ and $g_0$ and $L$ defined above.
New integrable systems

Let $L = L(u)$ be a Nijenhuis operator, $u = (u^1, \ldots, u^n)^\top$, $u^i = u^i(x, t)$, and $\chi_L(\lambda) = \det(\lambda \text{Id} - L(u)) = \lambda^n - \sigma_1(u)t^{n-1} - \cdots - \sigma_n(u)$.

Then the following system of PDEs is integrable:

$$u_t = \left( \frac{1}{\sqrt{\det(L - \lambda \text{Id})}} \right)_{xxx} \xi + \frac{1}{\sqrt{\det(L - \lambda \text{Id})}}(L - \lambda \text{Id})^{-1}u_x$$

where $\xi = \xi(u)$ is a vector field such that $\xi(\sigma_i) = m_i = \text{const.}$

This system admits infinitely many conservation laws and commuting symmetries.

Limit case for $\lambda \to \infty$:

$$u_t = (\text{tr } L)_{xxx} \xi + (L - \frac{1}{2}\text{tr } L) u_x$$

If $n = 1$, then $L = (u)$, $\text{tr } L = u$, $\xi = \frac{\partial}{\partial u}$. Then the system becomes

$$u_t = u_{xxx} + \frac{1}{2} uu_x,$$

which is exactly the KdV equation.
1. Start from a Nijenhuis operator $L$
2. Add some combinatorial data (special oriented graph + parameters)
3. Construct Nienhuis pencil $\mathcal{R} = \{R_c\}_{c \in \mathbb{R}^m}$, where $R_c = R(c, L)$
4. Flat pencil of metrics $\mathcal{G} = \mathcal{R} g_0 = \{R_c g_0\}_{c \in \mathbb{R}^m}$ with a special $g_0$
5. Main result 1: $\mathcal{G} = \{g_c = R_c g_0\}_{c \in \mathbb{R}^m}$ admits a dispersive perturbation, i.e., for each $g_c$ we can explicitly construct $h_c$ such that all the non-homogeneous Poisson brackets $\mathcal{B}_{h_c} + \mathcal{A}_{g_c}$ are pairwise compatible.
6. Main result 2: any 2-dim pencil $\{\lambda g^{ij} + \mu \bar{g}^{ij}\}$ admitting a dispersive perturbation is a subpencil of $\mathcal{G}$ (for suitable combinatorial data).
7. Chose a dispersive perturbation to get a pair of compatible Poisson brackets $\mathcal{B}_h + \mathcal{A}_g$ and $\mathcal{B}_{\bar{h}} + \mathcal{A}_{\bar{g}}$.
8. Construct and study relevant integrable PDE systems.
Multi-block Frobenius pencils

Step 1. Divide the coordinates \((x^1, \ldots, x^n)\) into \(B\) blocks:

\[
\begin{pmatrix}
(x_1^1, \ldots, x_{n_1}^1), & \ldots, & (x_B^1, \ldots, x_{n_B}^B)
\end{pmatrix}, \quad n_1 + \ldots + n_B = n. \tag{4}
\]

Step 2. On each \(\alpha\)-block define the so-called (covariant) Levi-Civita metrics \(g_{\alpha}^{\text{LC}}\) and \(n_{\alpha}\)-dimensional operators \(L_{\alpha}\):

\[
g_{\alpha}^{\text{LC}} = \sum_{s=1}^{n_{\alpha}} \prod_{j \neq s} \left(x_{\alpha}^s - x_{\alpha}^j\right) \left(d x_{\alpha}^s\right)^2, \quad L_{\alpha} = \text{diag}(x_1^\alpha, \ldots, x_{n_\alpha}^\alpha).
\]

Step 3. Consider a new block-diagonal metric \(g\)

\[
g = \text{diag}(g_1, \ldots, g_B) \quad \text{with} \quad g_{\alpha} = f_{\alpha}(x) \cdot g_{\alpha}^{\text{LC}},
\]

where \(f_{\alpha}(x)\) is a function defined by certain combinatorial data.

Step 4. Finally consider the pencil of (contravariant) metrics of the form \(\{Rg^{-1} \mid R \in \mathcal{R}\}\) where \(\mathcal{R}\) is the Nijenhuis pencil

\[
\mathcal{R} = \left\{ \text{diag}(P_1(L_1), P_2(L_2), \ldots, P_B(L_B)) \right\}, \quad \text{with} \quad \deg P_{\alpha} \leq n_{\alpha} + 1
\]

and some additional linear relations on \(P_{\alpha}\).
The functions $f_\alpha$ and relations on the coefficients of $P_\alpha$’s are determined by a combinatorial object, a graph $F$ with some special properties. This graph may consist of several connected components, each of which is a rooted tree whose edges are oriented from its leaves to the root, as shown:

Each vertex of $F$ is associated with a certain block of the above decomposition (4) and labelled by an integer number $\alpha \in \{1, \ldots, B\}$. In addition, to each edge $e_\alpha$ we assign a number $\lambda_\alpha$. 
Conditions of $f_\alpha$ and $P_\alpha$

Functions $f_\alpha$:
Take the oriented path from the vertex $\alpha$ to the root $\beta$:

$$\beta = \alpha_0 \overset{\lambda_{\alpha_1}}{\leftarrow} \alpha_1 \overset{\lambda_{\alpha_2}}{\leftarrow} \ldots \overset{\lambda_{\alpha_{k-1}}}{\leftarrow} \alpha_{k-1} \overset{\lambda_{\alpha_k}}{\leftarrow} \alpha_k = \alpha,$$

and set

$$f_\alpha = \prod_{i=1}^{k} \det(L_{\alpha_{i-1}} - \lambda_{\alpha_i} \cdot \text{Id}).$$

E.g. for $\alpha = 8$ in the above example: $1 \overset{\lambda_2}{\leftarrow} 2 \overset{\lambda_3}{\leftarrow} 3 \overset{\lambda_8}{\leftarrow} 8$, we have $f_8 = \det(L_1 - \lambda_2) \det(L_2 - \lambda_3) \det(L_3 - \lambda_8)$.

Polynomials $P_\alpha$:

If $(\lambda_\beta, \lambda_\gamma)$, then

1. $\lambda_\beta$ is a root of $P_\alpha$
2. $\lambda_\beta = \lambda_\gamma \Rightarrow \lambda_\beta$ is a double root of $P_\alpha$
3. $P_\beta(t) = (-1)^n P'(\lambda_\beta) t^{n_\beta + 1} + \ldots$
4. \ldots
We have constructed a family (pencil) of metrics

$$\{ \ R \ g^{-1} \ | \ R \in \mathcal{R} \ \} , \quad g = \sum_{\alpha} \ f_{\alpha}(x) \ g^{\text{LC}}_{\alpha},$$

(5)

where $\mathcal{R}$ is a Nijenhuis pencil that consists of operators of the form

$$R = \bigoplus_{\alpha} P_{\alpha}(L_{\alpha}) = \text{diag}(P_{1}(L_{1}), P_{2}(L_{2}), \ldots, P_{B}(L_{B})).$$

**Theorem**

The pencil (5) is Frobenius. In other words, all the (contravariant) metrics

$$\text{diag} \ (P_{1}(L_{1}) \ g_{1}^{-1}, \ldots, P_{B}(L_{B}) \ g_{B}^{-1}) \ \ \text{with} \ g_{\alpha} = f_{\alpha}(x) \cdot g^{\text{LC}}_{\alpha}$$

are flat, Poisson compatible and admit a common Frobenius coordinate system.

**Theorem**

Let $g$ and $\bar{g}$ be Poisson compatible (contravariant) flat metrics that admit a common Frobenius coordinate system. If the eigenvalues of

$$R = g \ \bar{g}^{-1}$$

are all different, then the pencil $\{ \lambda \ g^{ij} + \mu \ \bar{g}^{ij} \}$ is isomorphic to a 2-dim subpencil of (5) with suitable parameters.
Thank you for your attention

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