Lecture 4: Nijenhuis pencils and Compatible Poisson brackets

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Compatible structures

In mathematics, very often when working with a certain class $\mathcal{A}$ of objects, we want to distinguish those pairs $A, \widetilde{A} \in \mathcal{A}$ that are compatible in some sense. (This relation is reflexive and symmetric, but not necessarily transitive!)
Examples include:

1. compatible Poisson structures
2. geodesically equivalent metrics
3. flat metrics (in the context of Poisson brackets of hydrodynamic type)
4. compatible algebras defined by a certain property (Lie, associative, Frobenius)
5. compatible Nijenhuis operators

It is interesting that in cases 1, 4 and 5 the definition of compatibility is the same: $A \in \mathcal{A}$ and $\widetilde{A} \in \mathcal{A}$ are compatible if $A + \widetilde{A} \in \mathcal{A}$.
For instance, two associative algebras $(\mathfrak{a}, \star)$ and $(\mathfrak{a}, *)$ defined on the same vector space $\mathfrak{a}$ are compatible, if the new multiplication $\star + *$ defined by

$$\xi, \eta \in \mathfrak{a} \quad \mapsto \quad \xi \star \eta + \xi * \eta$$

is still associative.

Nijenhuis operators as partners of other geometric structures

- Consider two compatible Poisson structures P_1 and P_2, one of which is non-degenerate. Then the operator $L = P_2 P_1^{-1}$ is Nijenhuis.
- Consider two geodesically equivalent metrics g_1 and g_2. Then the operator $L = \left(\frac{\det g_2}{\det g_1}\right)^{\frac{1}{n+1}} g_2^{-1} g_1$ is Nijenhuis.
- Consider two compatible Poisson brackets of hydrodynamic type, determined by two flat (contravariant) metrics g_1 and g_2. Then the operator $L = g_1 g_2^{-1}$ is Nijenhuis.

For each of these situations it makes sense to introduce the following terminology:

- L is compatible with a Poisson structure P, if P and LP are compatible Poisson structures.
- L is geodesically compatible with g, if the metrics g and $\frac{1}{|\det L|} g L^{-1}$ are geodesically equivalent.
- L is Poisson compatible with a flat (contravariant) metric g, if Lg is flat and the metrics g and Lg define a pair of compatible Poisson brackets of hydrodynamic type.

Which Nijenhuis L can find a suitable partner?

Problem

Let L be a Nijenhuis operator. Does it admit

- *compatible Poisson structure P,*
- *geodesically compatible metric g,*
- *Poisson compatible (flat) metric g ?*

This question makes sense both in global and local context.

Comments.

- If the algebraic type of an operator L is locally constant (i.e. we consider a non-singular point), then the answer is essentially known in the first (Turiel) and second (V. Matveev, AB) cases.
- Thus the question is basically related to singular points: which singularities of the operator L are admissible in this context?
- If L is differentially non-degenerate at a given singular point, then the partner for L exists in each of the above cases.
- In the case of geodesic compatibility, all admissible singularities are know if g is positive definite (V. Matveev).

Pencils of compatible structures and Nijenhuis pencils

In each of the above mentioned cases:

- compatible Poisson brackets
- compatible flat metrics
- geodesically equivalent metrics

it makes sense to consider not just two, but a multi-parametric family $\mathcal{P}$ of pairwise compatible structures.
Since each pair of compatible objects is related by means of a Nijenhuis operator, together with $\mathcal{A}$ we obtain a natural family $\mathcal{L}$ of Nijenhuis operators:

$$\mathcal{P} = \{P_c\} \quad \mapsto \quad \mathcal{L} = \{L_c\}$$

Namely, we first choose a certain (non-degenerate) "reference point" $P \in \mathcal{P}$ and then take all L_c's that link P with all the other elements $P_c \in \mathcal{P}$.

Important observation: $\mathcal{L} = \{L_c\}$ is a vector space that consists of Nijenhuis operators (= Nijenhuis pencil).

Definition

Two Nijenhuis operators L_1 and L_2 are called compatible if their sum $L_1 + L_2$ is a Nijenhuis operator also.

Compatible Nijenhuis operators

Consider a pencil of compatible structures $\mathcal{P} = \{P_c\}_{c\in\mathbb{R}^m}$ (e.g. Poisson structures). As discussed, we can consider a natural Nijenhuis pencil associated to it:

$$\mathcal{L} = \{L_c\}_{c\in\mathbb{R}^n}, \quad \text{where } L_c = P_c P^{-1}$$

for some fixed $P \in \mathcal{P}$ so that $\mathcal{P} = \mathcal{L}P$. Notice that the choice of the "reference point" $P \in \mathcal{P}$ is not unique.

Take three compatible structures P_0, P_1 and P_2. Then $L_1 = P_1 P_0^{-1}$ and $L_2 = P_2 P_0^{-1}$ are compatible Nijenhuis operators. On the other hand, since P_1 and P_2 are compatible also, then the operator $P_1 P_2^{-1} = L_1 L_2^{-1}$ is Nijenhuis. This "observation" suggests that compatibility of two Nijenhuis operators L_1 and L_2 is somehow related to the fact that $L_1 L_2^{-1}$ is Nijenhuis. Indeed, we have

Lemma

Consider a pair of non-degenerate Nijenhuis operators L_1 and L_2. Then L_1 and L_2 are compatible if and only if $L_1 L_2^{-1}$ is Nijenhuis operator.

Proof

We need to analyse the Nijenhuis condition $\mathcal{N}_{L_1+L_2} = 0$. It is easy to see that $\mathcal{N}_{L_1+L_2} = \mathcal{N}_{L_1} + \mathcal{N}_{L_2} + \mathcal{N}_{L_1,L_2}$, where the last term (known as Frolicher-Nijenhuis bracket of L_1 and L_2) takes the form:

$$\mathcal{N}_{L_1,L_2}(v,w) = L_1L_2[v,w] + L_2L_1[v,w] + [L_1v, L_2w] + [L_2v, L_1w] - \\ - L_1[L_2v, w] - L_1[v, L_2w] - L_2[L_1v, w] - L_2[v, L_1w]$$

Let us prove the following identity, which immediately implies the statement of the lemma:

$$\mathcal{N}_{L_1}(v,w) + L_1L_2^{-1}L_1L_2^{-1}\mathcal{N}_{L_2}(v,w) - \mathcal{N}_{L_1L_2^{-1}}(L_2v, L_2w) = L_1L_2^{-1}\mathcal{N}_{L_1,L_2}(v,w).$$

We have

$$\mathcal{N}_{L_1}(v,w) + L_1L_2^{-1}L_1L_2^{-1}\mathcal{N}_{L_2}(v,w) - \mathcal{N}_{L_1L_2^{-1}}(L_2v, L_2w) =$$

$$= L_1L_2^{-1}L_2L_1[v,w] + [L_1v, L_1w] - L_1L_2^{-1}L_2[L_1v, w] - L_1L_2^{-1}L_2[v, L_1w] +$$

$$+ L_1L_2^{-1}L_1L_2[v,w] + L_1L_2^{-1}L_1L_2^{-1}[L_2v, L_2w] - L_1L_2^{-1}L_1[L_2v, w] - L_1L_2^{-1}L_1[v, L_2w]$$

$$- L_1L_2^{-1}L_1L_2^{-1}[L_2v, L_2w] - [L_1v, L_1w] + L_1L_2^{-1}[L_1v, L_2w] + L_1L_2^{-1}[L_2v, L_1w] =$$

$$= L_1L_2^{-1}\mathcal{N}_{L_1,L_2}(v,w),$$

as stated.

Pencils of Nijenhuis operators

Definition. A pencil of Nijenhuis operators is a vector subspace $\mathcal{L} \subset \mathrm{End}(TM)$, which consist of Nijenhuis operators.

Examples:

- all constant operators $\left\{A = \left(a^i_j\right),\ a^i_j \in \mathbb{R}\right\}$
- all scalar operators $\{A = f(x) \cdot \mathrm{Id},\ f \in C^\infty(M)\}$

A natural general question is description of examples and classification of Nijenhuis pencils (at generic points).

Maximal pencils are of particular interest (i.e. those which cannot be extended).

Examples:

- diagonal pencil $\left\{L = \mathrm{diag}\big(f_1(x_1), f_2(x_2), \ldots, f_n(x_n)\big)\right\}$
- pencil of operators of the form $\left\{L = \Big(a_{ij} + b_i x_j + x_i b_j + a x_i x_j\Big), a_{ij} = a_{ji}\right\}$

Since we do not know almost anything about Nijenhuis pencils, a natural task, at the moment, would be just finding and studying examples.

Poisson structures and Hamiltonian systems (see Lecture 3)

Definition

A *Poisson structure* is defined to be a skew-symmetric tensor P^{ij} of type $(2,0)$ such that the corresponding operation (*Poisson bracket*)

$$\{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) \ \to \ C^\infty(M),$$

$$f, g \ \mapsto \ \{f,g\} = P(\mathrm{d}\, f, \mathrm{d}\, g) = \sum P^{ij} \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j}$$

satisfies the Jacobi identity:

$$\{\{f,g\},h\} + \{\{g,h\},f\} + \{\{h,f\},g\} = 0 \quad \text{for all } f, g, h \in C^\infty(M).$$

Remark. If $\det P \neq 0$, then we can define the inverse tensor $\omega = P^{-1}$ which is a differential 2-form. The Jacobi identity, in this case, means exactly that $\mathrm{d}\,\omega = 0$, i.e., ω is a symplectic structure.

Definition

Let H be a smooth function. The vector field

$$\mathcal{X}_H = P(\mathrm{d}\, H, \cdot), \quad \text{(equivalently } \mathcal{X}_H^j = P^{ij} \tfrac{\partial H}{\partial x^i} \text{ or } \mathcal{X}_H = P\, \mathrm{d}\, H\text{)}$$

is said to be *Hamiltonian* (w.r.t. P and with the *Hamiltonian function* H).

Compatible Poisson structures and bi-Hamiltonian systems

Definition

Two Poisson structures P and $\widetilde{P}$ are called *compatible*, if their sum $P + \widetilde{P}$ is also a Poisson structure.

Definition

A vector field $\mathcal{X}$ is called *bi-Hamiltonian*, if it is Hamiltonian with respect to two compatible Poisson brackets P and $\widetilde{P}$, that is,

$$\mathcal{X} = P\,\mathrm{d}\,H = \widetilde{P}\,\mathrm{d}\,\widetilde{H}.$$

Question. Why are we interested in bi-Hamiltonian systems? What are their properties? And why are such systems "better" than the others?

We will assume here that at least one of the compatible Poisson structures P and $\widetilde{P}$ is non-degenerate so that e.g. P^{-1} makes sense.

Basic properties

Consider the linear operator $L = \widetilde{P}P^{-1}$ or, equivalently,

$$P(L^*\mathrm{d}\,f, \mathrm{d}\,g) = \widetilde{P}(\mathrm{d}\,f, \mathrm{d}\,g) \quad \text{for all } f, g \in C^\infty(M).$$

Proposition

Let P and $\widetilde{P}$ be Poisson structures. Then P and $\widetilde{P}$ are compatible if and only if $L = \widetilde{P}P^{-1}$ is a Nijenhuis operator.

In the theory of bi-Hamiltonian systems, the operator L plays an important role and is known as *recursion operator*.

In particular, notice the following formula for Hamiltonian vector fields:

$$\widetilde{\mathcal{X}}_f = L\,\mathcal{X}_f.$$

Also notice that without loss of generality we may, at least locally, assume that L is non-degenerate. Indeed, replacing $\widetilde{P}$ with $\widetilde{P} + \lambda P$ leads to $L \mapsto L + \lambda\,\mathrm{Id}$.

Proof

We first notice that the Jacobi identity for a Poisson structure P can be rewritten, in terms of Hamiltonian vector fields, as follows:

$$\mathcal{X}_{\{f,g\}} = [\mathcal{X}_f, \mathcal{X}_g].$$

Hence, the Jacobi identity for $P + \widetilde{P}$ can be rewritten as follows (below we use the fact that P and $\widetilde{P}$ are Poisson and remove the terms that vanish automatically due to this fact):

$$\widetilde{\mathcal{X}}_{\{f,g\}} + \mathcal{X}_{\widetilde{\{f,g\}}} = [\widetilde{\mathcal{X}_f}, \mathcal{X}_g] + [\mathcal{X}_f, \widetilde{\mathcal{X}_g}]$$

Now we use the fact that $\widetilde{\mathcal{X}_f} = L\mathcal{X}_f$ for any f:

$$L\mathcal{X}_{\{f,g\}} + L^{-1}\widetilde{\mathcal{X}}_{\widetilde{\{f,g\}}} = [L\mathcal{X}_f, \mathcal{X}_g] + [\mathcal{X}_f, L\mathcal{X}_g]$$

Finally we use again the fact that P and $\widetilde{P}$ are both Poisson to get

$$L[\mathcal{X}_f, \mathcal{X}_g] + L^{-1}[L\mathcal{X}_f, L\mathcal{X}_g] = [L\mathcal{X}_f, \mathcal{X}_g] + [\mathcal{X}_f, L\mathcal{X}_g]$$

Multiplying this identity by L we get the Nijenhuis relation for L. It remains to recall that P is non-degenerate so that Hamiltonian vector fields generate the whole tangent space.

Basic facts (continued...)

Proposition

Let $\mathcal{X}$ be a bi-Hamiltonian vector field w.r.t. P and $\widetilde{P}$. Then the coefficients σ_k of the characteristic polynomial of the recursion operator $L = \widetilde{P}P^{-1}$ are first integrals of $\mathcal{X}$. Moreover, these integrals commute with respect to the both structures P and $\widetilde{P}$.

Proof. 1. First integrals. It is a well known fact that P is preserved by each Hamiltonian vector field $\mathcal{X}_f$, i.e., $\mathcal{L}_{\mathcal{X}_f} P = 0$. Since in our case $\mathcal{X}$ is bi-Hamiltonian, we conclude that $\mathcal{X}$ preserves both P and $\widetilde{P}$.
Hence, $\mathcal{X}$ also preserves the recursion operator $L = \widetilde{P}P^{-1}$, and therefore all of its algebraic invariants such as coefficients σ_k of $\chi_L(t)$.
2. Commutativity. Instead of σ_k, we check commutativity of $f_k = \frac{1}{k}\operatorname{tr} L^k$, i.e., the identities $\{f_k, f_m\} = 0$. We use the following property of Nijenhuis operators $\mathrm{d}\, f_k = L^* \mathrm{d}\, f_{k-1}$ which implies:

$$\{f_k, f_m\} = P(\mathrm{d}\, f_k, \mathrm{d}\, f_m) = P(L^* \mathrm{d}\, f_{k-1}, \mathrm{d}\, f_m) = \widetilde{\{f_{k-1}, f_m\}} = \{f_{k-1}, f_{m+1}\}$$

and by induction $\{f_k, f_m\} = \widetilde{\{f_{k-s}, f_{m+s-1}\}} = \{f_{k-s}, f_{m+s}\}$. It remains to notice that for a suitable s we have either $k-s = m+s$ or $k-s = m+s-1$ and the statement follows from skew symmetry of P and $\widetilde{P}$.

Applications to ∞-dim Poisson Structures

Poisson structure, Hamiltonian $\quad\mapsto\quad$ Hamiltonian system of ODEs

$$H(x), P^{ij}(x) \quad\mapsto\quad \frac{\mathrm{d}\,x^i}{\mathrm{d}\,t} = P^{ij}\frac{\partial H}{\partial x^j}$$

∞-dimensional (differential-geomemetric) Poisson brackets, Hamiltonian functional (differential polynomial) in $u^1, \dots, u^n \quad\mapsto\quad$ Hamiltonian system of PDEs

$$H(u, u_x, \dots),\ \mathcal{P}^{ij} \quad\mapsto\quad \frac{\partial u^i}{\partial t} = \mathcal{P}^{ij}\left[\frac{\delta H}{\delta u^j}\right] = F^i(x, u, u_x, u_{xx}, \dots)$$

Darboux theorem: Poisson bracket can be reduces to constant form $P^{ij} = \text{const}$

∞-dimensional case: 'sometimes' Poisson bracket can be reduced to a constant form (so called Poisson-Darboux structures)

Examples:

$$\begin{aligned} \mathcal{P}^{ij} &= g^{ij}\,D \quad &\mapsto\quad & \mathcal{P} = \mathcal{A}_g = g^{ij}(u)\,D - \Gamma^{ij}_s\,u^s_x \\ \mathcal{P}^{ij} &= h^{ij}\,D^3 \quad &\mapsto\quad & \mathcal{P} = \mathcal{B}_h = h^{ij}(u)\,D^3 - 3\,\Gamma^{ij}_s\,u^s_x\,D^2 + \dots \end{aligned}$$

Statement of the problem

Recall that two Poisson brackets $\mathcal{P}_1$ and $\mathcal{P}_2$ are *compatible*, if the sum $\mathcal{P}_1 + \mathcal{P}_2$ is a Poisson bracket also.

- Problem 1. Describe pairs of flat metrics h, g such that $\mathcal{B}_h + \mathcal{A}_g$ is a Poisson bracket (equivalently, $\mathcal{B}_h$ and $\mathcal{A}_g$ are compatible).
- Problem 2. Describe compatible Poisson brackets of the form $\mathcal{B}_h + \mathcal{A}_g$ and $\mathcal{B}_{\bar{h}} + \mathcal{A}_{\bar{g}}$. Equivalently, describe quadruples of flat metrics g, $\bar{g}$, h, $\bar{h}$ such that ...
- Problem 2'. Describe pairs of flat metrics g and $\bar{g}$, for which we can find another pair of flat metrics h and $\bar{h}$ such that $\mathcal{B}_h + \mathcal{A}_g$ and $\mathcal{B}_{\bar{h}} + \mathcal{A}_{\bar{g}}$ are Poisson and compatible.
  This "procedure" $\mathcal{A}_g,\ \mathcal{A}_{\bar{g}} \mapsto \mathcal{B}_h + \mathcal{A}_g, \mathcal{B}_{\bar{h}} + \mathcal{A}_{\bar{g}}$ is known as dispersive perturbation.

Motivation. Integrability of many multicomponent PDE systems is related to bi-Hamiltonian structures of this kind. Examples include multicomponent KdV, Camassa-Holm and Harry Dym equations.

Relation to Nijenhuis pencils (see Lecture 3)

Definition (B. Dubrovin)

Two contravariant flat metrics g and $\bar{g}$ are Poisson compatible if:

1. $\hat{g}^{ij} = g^{ij} + \bar{g}^{ij}$ is flat,
2. $\hat{\Gamma}^{ij}_s = \Gamma^{ij}_s + \bar{\Gamma}^{ij}_s$.

This condition is equivalent to compatibility of the corresponding first order Poisson brackets $\mathcal{A}_g$ and $\mathcal{A}_{\bar{g}}$. One also says that $\{\lambda\, g^{ij} + \mu\, \bar{g}^{ij}\}$ is a flat two-dimensional pencil of (contravariant) metrics. In a similar way, we can define flat pencils $\mathcal{G} = \{g^{ij}_c\}_{c\in\mathbb{R}^n}$ of any dimension m.

Fact (E. Ferapontov)

If g and $\bar{g}$ are Poisson compatible, then $R = \bar{g}g^{-1}$ is a Nijenhuis operator.

Hence, every flat pencil can be written as $\mathcal{G} = \{R_c\, g\}_{c\in\mathbb{R}^m}$, where $\mathcal{R} = \{R_c\}_{c\in\mathbb{R}^m}$ is a Nijenhuis pencil.

Frobenius pencils

Definition

A commutative associative algebra $(\mathfrak{a}, \star)$ is called *Frobenius* if it is endowed with a nondegenerate symmetric bilinear form $b(\ ,\ )$ such that

$$b(\xi \star \eta, \zeta) = b(\xi, \eta \star \zeta), \quad \text{for all } \xi, \eta, \zeta \in \mathfrak{a}.$$

Remarkable property (A. Balinskii, S. Novikov)

Let $e^1, \dots, e^n$ be a basis of $\mathfrak{a}$ and $e^i \star e^j = \sum a^{ij}_s e^s$, $b(e^i, e^j) = b^{ij}$. Then the contravariant metric

$$g^{ij}(u) = b^{ij} + a^{ij}_s u^s \tag{1}$$

is flat.

Definition

A flat pencil $\mathcal{G} = \{g_c\}_{c \in \mathbb{R}^n}$ is called *Frobenius* if there exists a coordinate system $u^1, \dots, u^n$ such that every metric $g_c \in \mathcal{G}$ takes the form (1) for a certain Frobenius algebra $(\mathfrak{a}_c, b_c)$, $c \in \mathbb{R}^n$.

Solution of Problem 1

Theorem

Let g and h be two flat metrics. Then $\mathcal{B}_h + \mathcal{A}_g$ is Poisson if and only if there exists a coordinate system $u^1, \dots, u^n$ such that:

1. $g^{\alpha\beta}(u) = b^{\alpha\beta} + a_s^{\alpha\beta} u^s$, *and* $u^1, \dots, u^n$ *are Frobenius for* g;
2. $h^{\alpha\beta} = \text{const}$;
3. $h = \left(h^{\alpha\beta}\right)$ *is a Frobenius form for* $\mathfrak{a}$.

The coordinates $(u^1, \dots, u^n)$ are called *Frobenius* for $\mathcal{B}_h + \mathcal{A}_g$.

Frobenius coordinates are not unique: they are defined up to affine transformations.

Pre-solution of Problem 2 in terms of Frobenius pencils

Theorem

Consider two non-homogeneous Poisson structures $\mathcal{B}_h + \mathcal{A}_g$ and $\mathcal{B}_{\bar{h}} + \mathcal{A}_{\bar{g}}$ and suppose that $R = \bar{h}h^{-1}$ has n different eigenvalues.

They are compatible if and only if

(i) *the pencil $\{\lambda\, g^{ij} + \mu\, \bar{g}^{ij}\}$ is Frobenius, i.e., take form* (1) *in suitable coordinates $u^1, \dots, u^n$;*

(ii) *in these coordinates, $\lambda\, h^{ij} + \mu\, \bar{h}^{ij}$ are constant and define another Frobenius form for the Frobenius algebra related to $\lambda\, g^{ij} + \mu\, \bar{g}^{ij}$.*

Moreover, if $\{\lambda\, g^{ij} + \mu\, \bar{g}^{ij}\}$ is a Frobenius pencil, i.e., (i) *holds, then we can always find h and $\bar{h}$ satisfying* (ii).

Conclusion. The problem reduces to classification of two-dimensional Frobenius pencils.

AFF-pencil (Antonowicz–Fordy–Frobenius), see Lecture 3

Consider a real affine space $V \simeq \mathbb{R}^n$ with coordinates $u^1, \dots, u^n$ and define Nijenhuis operator L and contravariant metric g_0 on it by:

$$L = \begin{pmatrix} u^1 & 1 & 0 & \dots & 0 \\ u^2 & 0 & 1 & \dots & 0 \\ & & & \ddots & \\ u^{n-1} & 0 & 0 & \dots & 1 \\ u^n & 0 & 0 & \dots & 0 \end{pmatrix}, \quad g_0 = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 & 1 \\ 0 & 0 & \dots & 0 & 1 & -u^1 \\ 0 & 0 & \dots & 1 & -u^1 & -u^2 \\ & & \iddots & & & \\ 0 & 1 & \dots & -u^{n-4} & -u^{n-3} & -u^{n-2} \\ 1 & -u^1 & \dots & -u^{n-3} & -u^{n-2} & -u^{n-1} \end{pmatrix}.$$

Consider

$$\{g = P(L)g_0\}, \quad \text{where } P(\cdot) \text{ is a polynomial of degree} \leq n$$

This is a *Frobenius pencil*, i.e., $(u^1, \dots, u^n)$ are common Frobenius coordinates for all g's.
It is a remarkable fact that for each $g = P(L)g_0$ we can find a partner h such that $\mathcal{B}_h + \mathcal{A}_g$ is a Poisson structure and all of them are compatible (Antonowicz, Fordy):

$$g^{\alpha\beta} = b^{\alpha\beta} + a_s^{\alpha\beta} u^s \quad \mapsto \quad h^{\alpha\beta} = h^{\alpha\beta}(g, m) = m^0 \cdot b^{\alpha\beta} + a_s^{\alpha\beta} m^s$$

where $m^0, m^1, \dots, m^n$ are some constants.

AFF pencil (continued), see Lecture 3

As a result we obtain an $n+1$ dimensional pencil of non-homogeneous Poisson structures of the form

$$\mathcal{F}_m = \{\mathcal{B}_h + \mathcal{A}_g \mid \ g = P(L)g_0, h = h(g, m)\}$$

Here $P(\cdot)$ serves as a parameter of the bracket within the AFF-pencil, whereas m parametrises dispersive perturbations of this pencil.
AFF pencil in diagonal coordinates (Ferapontov–Pavlov): Consider the eigenvalues of L as local coordinates $x^1, \dots, x^n$. In these coordinates, g_0 and L become both diagonal:

$$g_0 = g_{\text{LC}} = \sum_{i=1}^{n} \left(\prod_{s \neq i} (x^i - x^s) \right)^{-1} \left(\tfrac{\partial}{\partial x^i}\right)^2, \quad L = \operatorname{diag}(x^1, ..., x^n), \qquad (2)$$

so that the AFF pencil becomes diagonal too:

$$\{P(L)g_{\text{LC}}\}, \quad \text{where } P(\cdot) \text{ is a polynomial of degree} \leq n. \qquad (3)$$

Transition from diagonal to Frobenius coordinates is natural: u_i are elementary symmetric polynomials in $x^1, \dots, x^n$.

Classification in the 'generic' case

Our goal is to classify pairs of compatible flat metrics g and $\bar{g}$ that admit a common Frobenius coordinate system (equivalently, Frobenius pencils).

Theorem

Let $\{\lambda g + \mu \bar{g}\}$ be a Frobenius pencil. Assume that the eigenvalues of the Nijenhuis operator $R = \bar{g} g^{-1}$ are all different and in the diagonal coordinates (such that R is diagonal) every diagonal component of g depends on all variables. Then this pencil is contained in the AFF-pencil, in other words, there exists a coordinate system $(x^1, \dots, x^n)$ such that

$$g = P(L) g_0 \quad \text{and} \quad \bar{g} = Q(L) g_0.$$

for some polynomials $P(\cdot)$ and $Q(\cdot)$ of degree $\leq n$ and g_0 and L defined above.

New integrable systems

Let $L = L(u)$ be a Nijenhuis operator, $u = (u^1, \dots, u^n)^\top$, $u^i = u^i(x,t)$, and $\chi_L(\lambda) = \det(\lambda \operatorname{Id} - L(u)) = \lambda^n - \sigma_1(u)t^{n-1} - \dots - \sigma_n(u)$.

Then the following system of PDEs is integrable:

$$\boxed{u_t = \left(\frac{1}{\sqrt{\det(L - \lambda \operatorname{Id})}}\right)_{xxx} \xi + \frac{1}{\sqrt{\det(L - \lambda \operatorname{Id})}} (L - \lambda \operatorname{Id})^{-1} u_x}$$

where $\xi = \xi(u)$ is a vector field such that $\xi(\sigma_i) = m_i = \text{const}$.

This system admits infinitely many conservation laws and commuting symmetries.

Limit case for $\lambda \to \infty$:

$$u_t = (\operatorname{tr} L)_{xxx}\, \xi + \left(L - \tfrac{1}{2}\operatorname{tr} L\right) u_x$$

If $n = 1$, then $L = (u)$, $\operatorname{tr} L = u$, $\xi = \frac{\partial}{\partial u}$. Then the system becomes

$$u_t = u_{xxx} + \tfrac{1}{2} u u_x,$$

which is exactly the KdV equation.

General case: Solution based on Nijenhuis Geometry

1. Start from a Nijenhuis operator L
2. Add some combinatorial data (special oriented graph + paremeters)
3. Construct Nienhuis pencil $\mathcal{R} = \{R_c\}_{c\in\mathbb{R}^m}$, where $R_c = R(c, L)$
4. Flat pencil of metrics $\mathcal{G} = \mathcal{R}g_0 = \{R_c\, g_0\}_{c\in\mathbb{R}^m}$ with a special g_0
5. Main result 1: $\mathcal{G} = \{g_c = R_c\, g_0\}_{c\in\mathbb{R}^m}$ admits a dispersive perturbation, i.e., for each g_c we can explicitly construct h_c such that all the non-homogeneous Poisson brackets $\mathcal{B}_{h_c} + \mathcal{A}_{g_c}$ are pairwise compatible.
6. Main result 2: any 2-dim pencil $\{\lambda\, g^{ij} + \mu\, \bar{g}^{ij}\}$ admitting a dispersive perturbation is a subpencil of $\mathcal{G}$ (for suitable combinatorial data).
7. Chose a dispersive perturbation to get a pair of compatible Poisson brackets $\mathcal{B}_h + \mathcal{A}_g$ and $\mathcal{B}_{\bar{h}} + \mathcal{A}_{\bar{g}}$.
8. Construct and study relevant integrable PDE systems.

Multi-block Frobenius pencils

Step 1. Divide the coordinates $(x^1, \dots, x^n)$ into B blocks:

$$\big(\underbrace{x_1^1, ..., x_1^{n_1}}_{x_1}, ..., \underbrace{x_B^1, ..., x_B^{n_B}}_{x_B}\big), \quad n_1 + ... + n_B = n. \tag{4}$$

Step 2. On each α-block define the so-called (covariant) Levi-Civita metrics g_α^{LC} and n_α-dimensional operators L_α:

$$g_\alpha^{\mathrm{LC}} = \sum_{s=1}^{n_\alpha} \prod_{j \neq s} (x_\alpha^s - x_\alpha^j)\, (\mathrm{d}\, x_\alpha^s)^2 \quad , \quad L_\alpha = \mathrm{diag}\big(x_\alpha^1, ..., x_\alpha^{n_\alpha}\big).$$

Step 3. Consider a new block-diagonal metric g

$$g = \mathrm{diag}(g_1, \dots, g_B) \quad \text{with } g_\alpha = f_\alpha(x) \cdot g_\alpha^{\mathrm{LC}},$$

where $f_\alpha(x)$ is a function defined by certain combinatorial data.

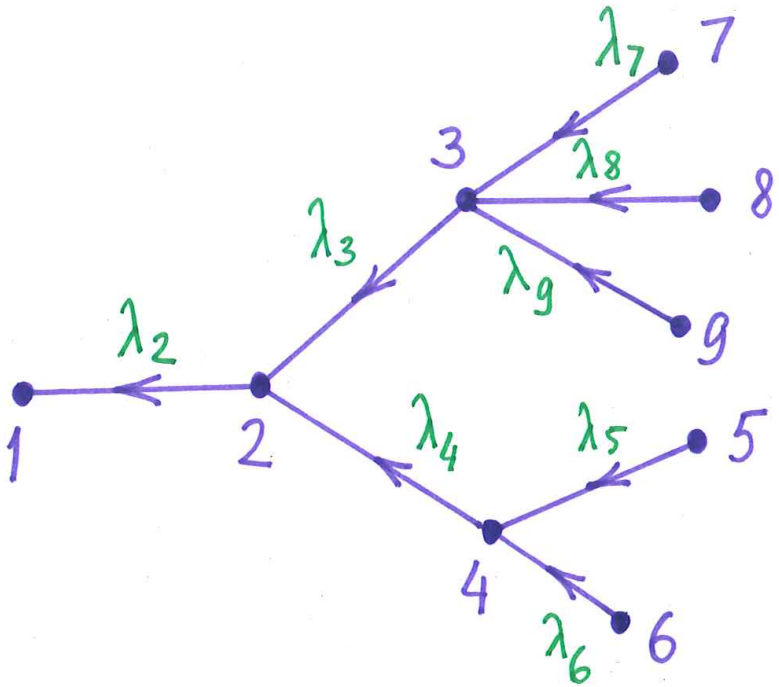
Step 4. Finally consider the pencil of (contravariant) metrics of the form $\big\{Rg^{-1} \mid R \in \mathcal{R}\big\}$ where $\mathcal{R}$ is the Nijenhuis pencil

$$\mathcal{R} = \big\{\mathrm{diag}\big(P_1(L_1), P_2(L_2), \dots, P_B(L_B)\big)\big\}, \text{with} \quad \deg P_\alpha \leq n_\alpha + 1$$

and some additional linear relations on P_α.

Combinatorial data

The functions f_α and relations on the coefficients of P_α's are determined by a combinatorial object, a graph F with some special properties. This graph may consists of several connected components, each of which is a rooted tree whose edges are oriented from its *leaves* to the *root*, as shown:



Each vertex of F is associated with a certain block of the above decomposition (4) and labelled by an integer number $\alpha \in \{1, ..., B\}$. In addition, to each edge e_α we assign a number λ_α.

Conditions of f_α and P_α

Functions f_α:

Take the oriented path from the vertex α to the root β:

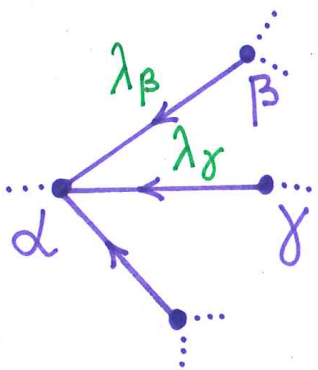
$$\beta = \alpha_0 \xleftarrow{\lambda_{\alpha_1}} \alpha_1 \xleftarrow{\lambda_{\alpha_2}} \dots \xleftarrow{\lambda_{\alpha_{k-1}}} \alpha_{k-1} \xleftarrow{\lambda_{\alpha_k}} \alpha_k = \alpha,$$

and set

$$f_\alpha = \prod_{i=1}^{k} \det(L_{\alpha_{i-1}} - \lambda_{\alpha_i} \cdot \mathrm{Id}).$$

E.g. for $\alpha = 8$ in the above example: $1 \xleftarrow{\lambda_2} 2 \xleftarrow{\lambda_3} 3 \xleftarrow{\lambda_8} 8$, we have $f_8 = \det(L_1 - \lambda_2)\det(L_2 - \lambda_3)\det(L_3 - \lambda_8)$.

Polynomials P_α:

If 

, then

(1) λ_β is a root of P_α

(2) $\lambda_\beta = \lambda_\gamma \Rightarrow \lambda_\beta$ is a double root of P_α

(3) $P_\beta(t) = (-1)^{n_\alpha} P'(\lambda_\beta)\, t^{n_\beta+1} + \dots$

(4) $\dots$

Main Theorems

We have constructed a family (pencil) of metrics

$$\left\{ R\, g^{-1} \mid R \in \mathcal{R} \right\}, \quad g = \sum_{\alpha} f_\alpha(x)\, g_\alpha^{\mathrm{LC}}, \tag{5}$$

where $\mathcal{R}$ is a Nijenhuis pencil that consists of operators of the form

$$R = \oplus_\alpha P_\alpha(L_\alpha) = \mathrm{diag}\big(P_1(L_1), P_2(L_2), \ldots, P_B(L_B)\big).$$

Theorem

The pencil (5) *is Frobenius. In other words, all the (contravariant) metrics*

$$\mathrm{diag}\left(P_1(L_1)\, g_1^{-1}, \ldots, P_B(L_B)\, g_B^{-1}\right) \quad \textit{with } g_\alpha = f_\alpha(x) \cdot g_\alpha^{\mathrm{LC}}$$

are flat, Poisson compatible and admit a common Frobenius coordinate system.

Theorem

Let g *and* $\bar{\mathrm{g}}$ *be Poisson compatible (contravariant) flat metrics that admit a common Frobenius coordinate system. If the eigenvalues of* $R = \mathrm{g}\,\bar{\mathrm{g}}^{-1}$ *are all different, then the pencil* $\left\{\lambda\, \mathrm{g}^{ij} + \mu\, \bar{\mathrm{g}}^{ij}\right\}$ *is isomorphic to a 2-dim subpencil of* (5) *with suitable parameters.*

Thank you for your attention

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