Character Sheaves.

Idea: Characters of $G(\mathbb{F}_q)$ come from sheaves on G.

Local Systems

A local system is a locally constant sheaf.
More precisely:

Def: A local system on a top. space X is a sheaf $\mathcal{F}$ s.t. for any $x \in X$ there is a nbh U of x with

$$\mathcal{F}|_U \cong \underline{\mathbb{C}}_U^{\oplus n} \quad \text{for some } n,$$

where $\underline{\mathbb{C}}_U$ is the constant sheaf.

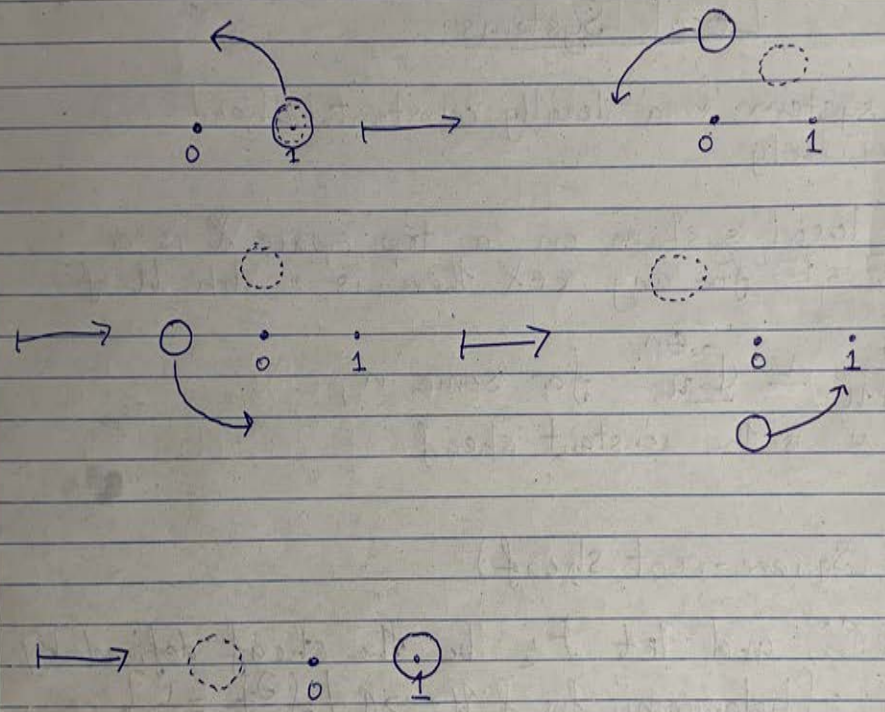
Example (Square-root sheaf):

Let $X = \mathbb{C}^*$, and let $\mathcal{F}_{\frac{1}{2}}$ be the sheaf defined by

$$\mathcal{F}_{\frac{1}{2}}(U) := \left\{ \text{holomorphic fns } f : U \to \mathbb{C} \;\middle|\; f \frac{\partial f}{\partial z} = \frac{1}{2} \right\}.$$

We can think of this as the sheaf of square-root functions.
$\mathcal{F}_{\frac{1}{2}}$ has no global sections, since $z^{\frac{1}{2}}$ cannot be defined globally.
But if $U \subset X$ does not loop around 0, then
$\mathcal{F}_{\frac{1}{2}}|_U \cong \underline{\mathbb{C}}_U$

Monodromy: Take U a nbh of $1 \in \mathbb{C}^*$ and $f \in \mathcal{F}_{\frac{1}{2}}(U)$. We can "analytically continue" f in a loop around the origin.



We end up with $-f \in \mathcal{F}_{\frac{1}{2}}(U)$

This gives us an action of $\pi_1(X)$ on the stalk $(\mathcal{F}_{\frac{1}{2}})_x \cong \mathbb{C}$ for any $x \in X$ (in this case $x = 1$).

This is called the monodromy representation

Now let $\Bbbk$ be any field

Let $\mathrm{Loc}(X, \Bbbk)$ be the category of ~~local~~ local systems on X with coefficients in $\Bbbk$.

Thm: We have an equivalence of categories

$$\mathrm{Mon}_x : \mathrm{Loc}(X, \Bbbk) \xrightarrow{\sim} \Bbbk[\pi_1(X, x)]\text{-mod},$$

given by taking the monodromy rep'n at $x \in X$.

Example (local systems on $X = \mathbb{C}^*$): (Again let $\Bbbk = \mathbb{C}^*$) ← not necessary

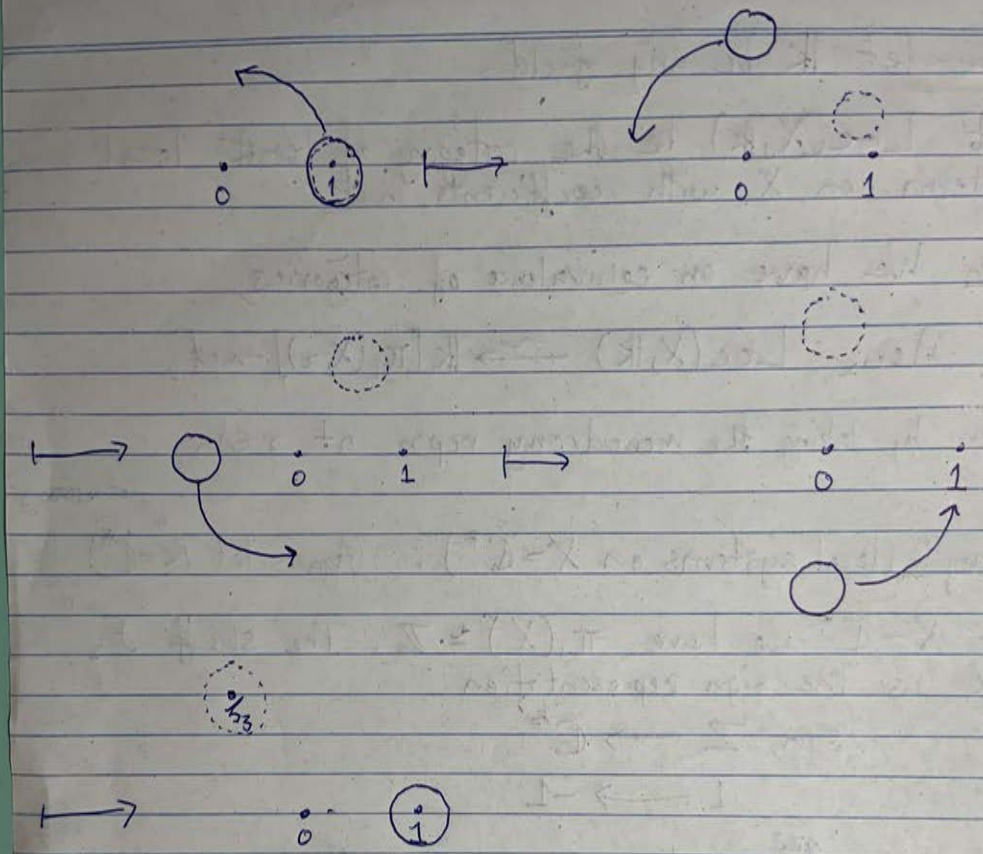
For $X = \mathbb{C}^*$ we have $\pi_1(X) \cong \mathbb{Z}$. The sheaf $\mathcal{F}_{\frac{1}{2}}$ gave us the sign representation

$$\mathrm{sgn} : \mathbb{Z} \longrightarrow \mathbb{C}^*$$
$$1 \longmapsto -1.$$

Let $\zeta_n = e^{\frac{2\pi i}{n}}$, and let $\rho_n : \mathbb{Z} \longrightarrow \mathbb{C}^*$ be
$1 \longmapsto \zeta_n$.

Then the corresponding local system $\mathcal{F}_{\frac{1}{n}}$ is the "n^{th} root sheaf".

E.g. take $n = 3$. Then monodromy acts as follows on $\mathcal{F}_{\frac{1}{3}}$, the 3rd-root sheaf:



i.e. monodromy gives ρ_3.

Kummer Local Systems: We have a correspondence between local systems and reps of $\pi_1(X)$. This correspondence is compatible with the tensor product, i.e.

$$Mon_x(\mathcal{F}\otimes\mathcal{G}) \simeq Mon_x(\mathcal{F})\otimes Mon_x(\mathcal{G})$$

for local systems $\mathcal{F}$ and $\mathcal{G}$.

For example, $\mathcal{F}_{\frac{1}{2}}\otimes\mathcal{F}_{\frac{1}{2}} \simeq \underline{\mathbb{C}}_X$.

As another example, $\mathcal{F}_{\frac{1}{3}}\otimes\mathcal{F}_{\frac{1}{3}} \simeq \mathcal{F}_{\frac{2}{3}}$, where $\mathcal{F}_{\frac{2}{3}}$ is the "$z^{\frac{2}{3}}$ sheaf".

Moreover, $\mathcal{F}_{\frac{1}{3}}\otimes\mathcal{F}_{\frac{1}{3}}\otimes\mathcal{F}_{\frac{1}{3}} \simeq \underline{\mathbb{C}}_X$.

Def: A Kummer local system on X is a local system $\mathcal{L}$ such that

$$\mathcal{L}^{\otimes n} \simeq \underline{\Bbbk}_X \text{ for some } n \text{ (with } (n, \operatorname{char}\Bbbk)=1).$$

For example, $\mathcal{F}_{\frac{1}{n}}$ is Kummer, since

$$\mathcal{F}_{\frac{1}{n}}^{\otimes n} \simeq \underline{\mathbb{C}}_X$$

In general, (for $\operatorname{char}\Bbbk = 0$) let $\mathcal{F}_{\frac{m}{n}}$ be the "$z^{\frac{m}{n}}$-sheaf" on $X=\mathbb{C}^*$. Then

$$\left(\mathcal{F}_{\frac{m}{n}}\right)^{\otimes n} \simeq \underline{\Bbbk}_X,$$

so $\mathcal{F}_{\frac{m}{n}}$ is Kummer.

Moreover, we have $\mathcal{F}_{\frac{m}{n}} \simeq \mathcal{F}_{\frac{m+n}{n}}$, since both correspond to the rep'n of $\pi_1(X) = \mathbb{Z}$ given by

$$\rho : \mathbb{Z} \longrightarrow \mathbb{C}^*$$
$$1 \longmapsto e^{2\pi i \left(\frac{m+n}{n}\right)} = e^{2\pi i \cdot \frac{m}{n}}.$$

(I am being a bit loosy-goosy with k and $\mathbb{C}$, but easy to generalize this for $\operatorname{char} k = 0$).

So for k characteristic 0, we have that Kummer local systems on $X = k^*$ are parametrized by $\mathbb{Q}/\mathbb{Z}$.

Now we generalize beyond $X = k^*$ to a split torus T (still in characteristic 0 for now). We have that Kummer local systems on T are ~~given by parametrized by pairs in~~ parametrized by

$$\left(X(T) \otimes_{\mathbb{Z}} \mathbb{Q}\right) / X(T),$$

i.e. if T is of rank k, then these are k-tuples of fractions up to equivalence relation

$$\left(\frac{m_1}{n_1}, \dots, \frac{m_i}{n_i}, \dots, \frac{m_k}{n_k}\right) \sim \left(\frac{m_1}{n_1}, \dots, \frac{m_i+n_i}{n_i}, \dots, \frac{m_k}{n_k}\right)$$

So essentially to specify a Kummer local system on T, we just pick k elements of $\mathbb{Q}/\mathbb{Z}$.

~~Now we move to~~ $q = p^k$

Now we fix notation. Let $k = \overline{\mathbb{F}_q}$, ~~$\overline{\mathbb{Q}_p}$~~
$E = \overline{\mathbb{Q}_\ell}$ for $(q, \ell) = 1$.

We will take sheaves on $G(\overline{\mathbb{F}_q})$ with values in E.

Then Kummer local systems on T are parametrized by

$$\hat{X} := (X(T) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)})/X(T),$$

i.e. as before, but we can't have p in the denominators
E.g. we ~~can~~ can't have "p^{th}-root sheaf".

Frobenius Invariance:

We have a Frobenius map $Fr: T \longrightarrow T$.
This induces a pull-back map ~~$Fr^*: Loc(X, E) \longrightarrow$~~ Lo
Fr^* on local systems on T.

Question: Which Kummer local systems (KLSs)
are invariant under Fr^*?

Exercise: If $T \simeq \mathbb{G}_m$ then Fr^*-invariant ~~local sy~~
KLSs on T correspond to $\frac{n}{q-1} \in \hat{X}$ as n varies,
i.e. think $\mathbb{Z}^{\frac{n}{q-1}}$-sheaf. So there are $q-1$ of them.
In general, Fr^*-invariant KLSs on T correspond to elements $\left(\frac{n_1}{q-1}, \dots, \frac{n_k}{q-1}\right) \in \hat{X}$.

Notice that we have a bijection

$$\{Fr^*\text{-invariant KLS's on } T\} \longleftrightarrow \text{Hom}(T(\mathbb{F}_q), E^*).$$

Given an Fr^*-invariant KLS on T, how do we recover the corresponding character on $T(\mathbb{F}_q)$?

Take an Fr^* invariant KLS $\mathcal{L}$ on any space X, defined over $\mathbb{F}_q$. This means we have an isomorphism $\phi: Fr^*\mathcal{L} \xrightarrow{\sim} \mathcal{L}$.
We want to get a fn $X(\mathbb{F}_q) \longrightarrow E^*$. Take $x \in X(\mathbb{F}_q)$, i.e. $Fr(x) = x$. We have

$$\mathcal{L}_x \xrightarrow{\phi^{-1}} (Fr^*\mathcal{L})_x = \mathcal{L}_{Fr(x)} = \mathcal{L}_x,$$

inducing $H^i_x(\mathcal{L}) \longrightarrow H^i_x(\mathcal{L})$. (If $\mathcal{L}$ is a complex)

Set $\chi_{\mathcal{L},\phi}(x) := \sum_i (-1)^i \text{Tr}(\phi, H^i_x(\mathcal{L})) \in E$.

This is how sheaves (or complexes of sheaves) on X lead to characters of $X(\mathbb{F}_q)$.

In particular for $X = T$, this gives the correspondence above btw Fr^*-invariant KLSs on T and $\text{Hom}(T(\mathbb{F}_q), E^*)$.

So we have a nice theory on the torus.
Can we extend this to general G?

Principal Series Representations:

Let $T \subset G$ be the split torus, and take a character

$$\lambda : T(\mathbb{F}_q) \longrightarrow \bar{E}^*$$

We can take the usual induction $\mathrm{Ind}_{B(\mathbb{F}_q)}^{G(\mathbb{F}_q)} \lambda$.

Given such a λ, we can also get an Fr^*-invariant local system $\mathcal{L} := \mathcal{L}_\lambda$ on T.

Then, consider the following maps:

$$\begin{array}{ccc} \tilde{G} & \xrightarrow{h} & B/[B,B] \cong T \\ \downarrow \pi & & \\ G & & \end{array}$$

Here, $\tilde{G} := \{(x, B) \in G \times \mathcal{B} \mid x \in B\}$ for $\mathcal{B}$ the flag variety. The maps h and π are then given by taking projections.

Now, we can pull-back, push-forward $\mathcal{L}$ to get

$$K_{\mathcal{L}} := \pi_* h^* \mathcal{L}.$$

Note that π_* is the derived push-forward here, so that $K_{\mathcal{L}}$ is a complex of sheaves on G. In fact $K_{\mathcal{L}}$ lies in the category of perverse sheaves on G.

Lazy Def: A perverse sheaf on X is a complex in the bounded derived category $D^b_c(X)$ of constructible sheaves on X, satisfying certain conditions on supp/H^i's.

We won't go into details here, all we need to know is that they are complexes of sheaves on (in our case) G.

Exercise: $\chi_{K_{\mathcal{L}},\phi} = \mathrm{Ind}_{B(\mathbb{F}_q)}^{G(\mathbb{F}_q)} \lambda$.

So our principal series comes from sheaves on G.

How do we get the other characters of $G(\mathbb{F}_q)$?

The trace formula $\chi_{K_{\mathcal{L}},\phi}$ used an isomorphism $\phi : K_{\mathcal{L}} \xrightarrow{\sim} \mathrm{Fr}^* K_{\mathcal{L}}$.
We can use other isomorphisms to get other characters:

Lemma: $\dim \mathrm{Hom}(K_{\mathcal{L}}, K_{\mathcal{L}'}) = |\{\omega \in W \mid \omega(\mathcal{L}) = \mathcal{L}'\}|$

Therefore if we take $\mathcal{L}$ with $\mathrm{Fr}^*\mathcal{L} \simeq \omega(\mathcal{L})$, we get an iso (if $\mathcal{L}$ is generic, i.e. $\mathrm{Stab}(\mathcal{L}) = e$):

$$\phi : K_{\mathcal{L}} \xrightarrow{\sim} \mathrm{Fr}^* K_{\mathcal{L}} = K_{\mathrm{Fr}^*\mathcal{L}} = K_{\omega(\mathcal{L})}$$

Computing $\chi_{K_{\mathcal{L}},\phi}$ will give us new irreducible characters.

If $\mathcal{L}$ is not generic, i.e. $\mathcal{L}$ has non-trivial stabilizer in W, then $\chi_{K_{\mathcal{L}},\phi}$ is in general not an irreducible character. This is where interesting behaviour happens.

Example (SL_2):

Let us first recall the DL parametrization of irreps of $SL_2(\mathbb{F}_q)$.

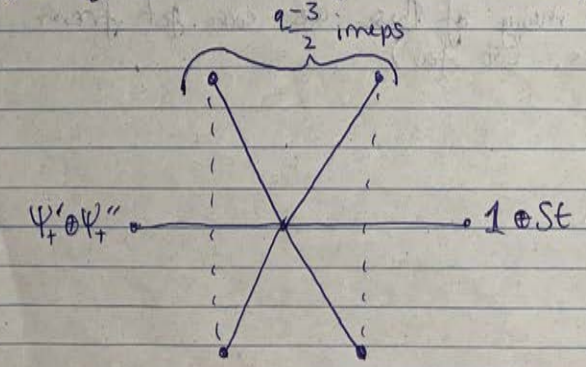
We first have principal series representations. We take a character λ of the split-torus $T(\mathbb{F}_q) \simeq \mathbb{F}_q^\times$ and induce.
We will let q be odd.

If λ is generic (i.e. $s(\lambda) \neq \lambda$ for $s \in W$) then $\mathrm{Ind}_{B(\mathbb{F}_q)}^{G(\mathbb{F}_q)} \lambda$ is an irrep. Moreover $\mathrm{Ind}\, s(\lambda) \simeq \mathrm{Ind}\, \lambda$.

If $\lambda = \mathbb{1}$ then $\mathrm{Ind}\, \lambda \simeq \mathbb{1}_G \oplus St_G$.

If $\lambda = \mathrm{sgn}$, then $\mathrm{Ind}\, \lambda \simeq \psi_+' \oplus \psi_+''$, where ψ_+', ψ_+'' are irreps of dim $\frac{q+1}{2}$.

Picture for $q=7$: (representing characters of $\mathbb{F}_q^*$ by roots μ_{q-1} of 1)

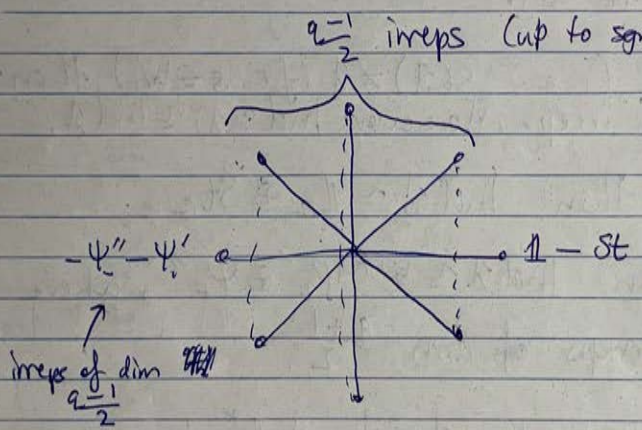


Then we have cuspidal characters.

Let $T_{ns} \cong \mu_{q+1}$ be the non-split torus.

We take a character λ on $T_{ns} \cong \mu_{q+1}$ and do the DL dance. This gives us virtual characters.

Picture for $q=7$:



Let's see how many of these we can get from character sheaves. (so far)

Character Sheaves on SL_2:

- **Principal Series:**

Recall our picture:

$$\begin{array}{ccc} \hat{G} & \xrightarrow{h} & B/[B,B] \cong T \\ \pi \downarrow & & \\ G & & \end{array}$$

Recall also that given a character $\lambda : T(\mathbb{F}_q) \longrightarrow E^*$ we get a local system $\mathcal{L} = \mathcal{L}_\lambda$ on T, and a perverse sheaf $K_{\mathcal{L}}$ on G via

$$K_{\mathcal{L}} := \pi_* h^* \mathcal{L}.$$

For $G = SL_2$, then we have $T(\mathbb{F}_q) \cong \mathbb{F}_q^*$, so characters of $T(\mathbb{F}_q)$ are given by $(q-1)^{\text{th}}$ roots of unity.

We saw earlier that

$$\chi_{K_\mathcal{L}, \phi} = \mathrm{Ind}_{B(\mathbb{F}_q)}^{G(\mathbb{F}_q)} \lambda.$$

When $\lambda = 1_T$, $K_{\mathcal{L}}$ splits into two irreducible perverse sheaves, with characters 1 and St.

When λ is a $(q-1)^{\text{th}}$ root of unity $\neq \pm 1$, then $K_{\mathcal{L}}$ is irreducible.

When $\lambda = -1$, however, something weird happens.
We know that

$$\chi_{K_\alpha, \varnothing} = \mathrm{Ind}_{B(\mathbb{F}_q)}^{G(\mathbb{F}_q)} \lambda = \psi_+' \oplus \psi_+'',$$

and K_α does split into two irreducible perverse sheaves $\mathcal{F} \oplus \mathcal{G}$, but these do not correspond to ψ_+' and ψ_+''.
In fact, taking χ on $\mathcal{F}$ and $\mathcal{G}$ does not produce irreducible characters.

Instead we get

$$\chi_{\mathcal{F}} = \frac{1}{2}(\psi_+' + \psi_+'' + \psi_-' + \psi_-''),$$

$$\chi_{\mathcal{G}} = \frac{1}{2}(\psi_+' + \psi_+'' - \psi_-' - \psi_-'').$$

So, if we want character sheaves to give a basis for $\mathrm{Irr}(G(\mathbb{F}_q))$, we will need to find other sheaves which give us different linear combinations of $\psi_+', \psi_+'', \psi_-', \psi_-''$.

For now, let us move to local systems on T with $\mathrm{Fr}^*(\mathcal{L}) = s(\mathcal{L})$ for $s \in W$.

Exercise: These are the "$(q+1)^{\text{th}}$-root" sheaves, i.e. we have 1 for every $(q+1)^{\text{th}}$ root of unity, $\lambda : \mu_{q+1} \longrightarrow \bar{E}^*$.
When $\lambda = \pm 1$, we have already found the corresponding character sheaves.
For $\lambda \neq \pm 1$, we recover the "cuspidal reps" obtained by DL induction on the non-split torus.

Question: Where are the missing sheaves?

Character Sheaves

What we have just seen is a particular case of the following construction.

Let $G_\omega = B\omega B$ for $\omega \in W$.

Then $G_\omega = T \times (\text{affine space})$ so we have a projection $pr: G_\omega \to T$.

Given a Kummer local system $\mathcal{L}$ on T we can take $pr^*\mathcal{L}$ on G_ω.

Then we can take the "IC extension" of $pr^*\mathcal{L}$ to all of G. We call this $A_{\mathcal{L},\omega}$.

Fact: If $\omega(\mathcal{L}) = \mathcal{L}$, then $A_{\mathcal{L},\omega}$ is B-equivariant w.r.t. conjugation

We want our character sheaves to be G-equivariant, since they need to yield class functions which are invariant on conjugacy classes.

Given a B-equivariant sheaf $\mathcal{F}$ on any space X, there is a procedure to produce a G-equivariant sheaf $\Gamma_B^G(\mathcal{F})$ on X:

We take the diagram

$$\begin{array}{ccccc} G\times X & \xrightarrow{\pi} & G\overset{B}{\times} X & \xrightarrow{m} & X \\ \downarrow{\scriptstyle pr} & & & & \\ X & & & & \end{array}$$

Then $pr^*\mathcal{F}$ is B-equivariant on $G\times X$, so becomes a sheaf $\tilde{\mathcal{F}}$ on $G\overset{B}{\times}X$.

Def: $\Gamma_B^G(X) := m_*\tilde{\mathcal{F}}$

Thus, we can take our B-equivariant sheaf $A_{\mathcal{L},\omega}$ on G and get a G-equivariant $\Gamma_B^G(A_{\mathcal{L},\omega})$ on G.

Def: Character sheaves are irreducible perverse sheaves that show up in the various $\Gamma_B^G(A_{\mathcal{L},\omega})$, for $\omega(\mathcal{L}) = \mathcal{L}$.

Rmk: The procedure we did on SL_2 was the case of $\omega = e$.

Exercise: Explain to Tasman why.

~~Thus, our missing sheaves come from the case $\omega = s$.~~

Thm: The characters produced in this way (by taking χ) form a basis of ~~$\mathrm{Irr}(G(\mathbb{F}_q))$~~ class functions on $G(\mathbb{F}_q)$.

For SL_2, we had characters

$$\chi_F = \frac{1}{2}(\psi_+' + \psi_+'' + \psi_-' + \psi_-''),$$

$$\chi_G = \frac{1}{2}(\psi_+' + \psi_+'' - \psi_-' - \psi_-'').$$

From taking $\Gamma_B^G(A_{\mathcal{L},\omega})$ for $\omega = s$, $\omega(\mathcal{L}) = \mathcal{L}$, we can get sheaves F', G' with characters

$$\chi_{F'} = \frac{1}{2}(\psi_+' - \psi_+'' + \psi_-' - \psi_-'')$$

$$\chi_{G'} = \frac{1}{2}(\psi_+' - \psi_+'' - \psi_-' + \psi_-'')$$

Thus we can obtain all characters ψ_+', ψ_+'', ψ_-', ψ_-'' via linear combinations of these.