$X \xrightarrow{f} Y \qquad H^*(Y, \mathbb{Q}_\ell) \to H^*(X, \mathbb{Q}_\ell)$

ring morphism

$H^*(X, \mathbb{Q}_\ell)$ finite dimensional.

Idea: Turns spaces (complicated) into linear algebra (finite dim vector spaces) (easier).

We have interesting actions of finite groups on spaces that only exist in pos characteristic, so linearising these spaces gives interesting representations.

$X \rightsquigarrow H^*(X, \mathbb{Q}_\ell)$ cohomology, contravariant functoriality

dual: $H_*(X, \mathbb{Q}_\ell)$

$\rightsquigarrow H_c^*(X, \mathbb{Q}_\ell)$ compactly supported cohomology, functorality is different.

dual: $H_*^{BM}(X, \mathbb{Q}_\ell)$

Notation:

$$X \rightsquigarrow \underline{X} := \bigoplus_i H^i_c(X, \mathbb{Q}_\ell)$$

$$|X| := \sum (-1)^i [H^i_c(X, \mathbb{Q}_\ell)]$$

If G acts on X, then it acts on $\underline{X}$, and $|X|$ lives in the Groth group of representations of G.

Example: $S^2 \longrightarrow \overset{0}{\mathbb{1}} \oplus \overset{1}{0} \oplus \overset{2}{\mathbb{1}} = 2$

$$\underset{\substack{\tau,\\ \text{antipodal}}}{\overset{\circlearrowleft}{S^2}} \longrightarrow \mathbb{1} \oplus 0 \oplus \sigma$$

$$\underset{\text{reflect}}{\overset{\circlearrowleft}{S^2}} \longrightarrow \mathbb{1} \oplus 0 \oplus \sigma$$

Have this for all primes ℓ, but we will fix one and ignore it from here.

$$X \rightsquigarrow H^*_c(X)$$

finite dim graded vector space

$$\begin{array}{c} X \\ \circlearrowleft \\ G \end{array} \rightsquigarrow \begin{array}{c} H^*_c(X) \\ \circlearrowleft \\ G \end{array}$$

In char p, have ~~Fro~~ Frobenius act F. $\rightsquigarrow$ F acts on $H^*_c(X)$.

~~Fix points of g on X~~

helpful to write $|X|$ as this complex of vector spaces, with G action, viewed in Groth group of RepG.

$$\boxed{\begin{array}{l} |X| = \text{class of } (-1)^i H^i_c(X) \\ \text{in } K_0(\text{Rep}G). \end{array}}$$

1. Each $H^i_c(X)$ is finite, so

$|X|$ is a virtual representation of G.

2. $|X \times \mathbb{A}^n| = |X|$, and $|*| = \mathbb{1}$
trivial rep.

$$X \xrightarrow{f} Y$$
all fibres $\cong \mathbb{A}^n$, then

$|X| = |Y|$ as reps of G.

3. $X = \coprod_{i=1}^{n} Z_i$ $\quad Z_i$ locally closed, then

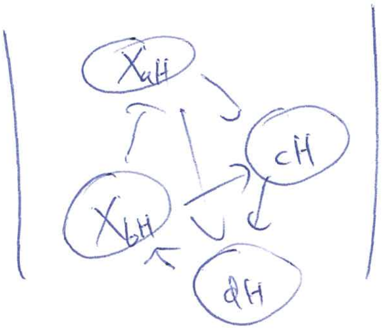
$$|X| = \sum_{i=1}^{n} |Z_i|$$

(Additive)

(and if $Z_i^\theta = 0$ for all θ irr of G, then $X^\theta = 0$)

4. $\left| G \times_H X \right| = I_H^G |X|$ if H acts on X.

$\parallel$



5. X affine, $\underline{X}/G = \underline{X/G} = \underline{X}^G$

$=$ isotypic comp of $\mathbb{1}$ in $\underline{X}$.

$F \circlearrowright X \circlearrowleft G$ commute, then

F acts on X/G, and trace

of F is $\frac{1}{G} \sum_{g \in G} \mathrm{Tr}(Fg, \underline{X})$

$\parallel$

$\mathrm{Tr}(F, \underline{X/G})$

6. $\underline{X \times Y} = \underline{X} \otimes \underline{Y}$

so $|X| \cdot |Y| = |X \times Y|$ in $K^0(\mathrm{Rep}\,G)$.

7. Trace of g on $\underline{X}$ is

$|X^g|_{ch}$ if X finite set, ~~the~~ holds in gen if g semisimple.

8. G connected acts on X, then acts trivially on $\underline{X}$.

9. Trace of g on $\underline{X}$ equals

$=$ trace of g on $\underline{X^s}$ $\quad g = su$

$=$ trace of u on $\underline{X^s}$.

~~10. 10. Trace of~~

$$k[t] \xrightarrow{T} k[t]$$

$$t \longmapsto t+1$$

Then T acts freely on $\mathbb{A}^1$, but trivial action on $\underline{\mathbb{A}^1}$, which is one dimensional in degree 2.

$$\mathbb{A}^1 \longrightarrow \mathbb{A}^1/T \cong \mathbb{A}^1$$

via

$$t^p - t \longleftarrow\!\shortmid\, t$$

This phenomenon is why étale cohomology works in our context.

$G \xrightarrow{\mathcal{L}} G$ is a covering space with deck group $G(\mathbb{F}_q)$.

The ~~F~~Frobenius map.

$X/_{\mathbb{F}_q}$ has a frobenius automorphism F

of $H_c^*(X, \mathbb{Q}_\ell)$.

Theorem: (Grothendieck-Lefschetz trace formula):

The graded trace of F on $\underline{X}$

equals the ~~[illegible]~~ cardinality $|X(\mathbb{F}_q)|$.

F a Frobenius on X, X^F

Ex: Give an automorphism of a top space without fixed points with nonzero trace on $H_c^*(X)$

~~This allows one also~~ (Intuition, Frob has all orbits finite) ~~[illegible]~~

Thm (Weil conjectures; Riemann hypothesis): X smooth proj over $\mathbb{F}_q$.

then the ~~[illegible]~~ eigenvalues of F on

$H_c^i(X, \mathbb{Q}_\ell)$ have $\underline{\text{absolute value}}$ $q^{i/2}$

(under any isomorphism $\overline{\mathbb{Q}_\ell} \xrightarrow{\sim} \mathbb{C}$).

Thm: $|H_{et}^i(X, \mathbb{Q}_\ell)| = |H^i(X \otimes \mathbb{Q})|$ if X defined over $\mathbb{Z}$.

Cor: If $X_{/\mathbb{F}_q}$ smooth, projective, the numbers $|X(\mathbb{F}_{q^n})|$ determine the dimensions of $H^i_c(X, \mathbb{Q}_\ell)$. (The Betti numbers)

Exercise, prove this. (Linear algebra).

Cor: If $X_{/\mathbb{F}_q}$ smooth, projective, and

$$|X(\mathbb{F}_{q^n})| = P(q^n) \text{ for some polynomial,}$$

then $P(q^n) = \sum_i (q^n)^i H^{2i}_c(X, \mathbb{Q}_\ell)$

is the poincare poly of X.

Exercise, compute the Betti numbers cohomology of the following spaces by counting points:

$\mathbb{A}^n$, $\mathbb{P}^n$, Flag variety, $E = V(X^3+Y^3+Z^3)$, $\mathrm{Pic}(C)$ C sm. proj curve.

$G(2,4)$ char $\neq 3$ (HARD)   (Riemann Roch, HARD?)

Thm: Trace of g on $\underline{X} = \bigoplus_i H^i_c(X, \mathbb{Q}_\ell)$

is given by

$$\mathrm{Tr}_g(\underline{X}) := \mathcal{L}(g, X) = \lim_{t \to \infty} R(t)$$

($X/\mathbb{F}_q$, g automorphism of finite order, assume F, g commute)

Where $R(t)$ is the rational function with power series expansion

$$R(t) := -\sum_{n=1}^{\infty} \left| X^{F^n g^{-1}} \right| t^n$$

where F is the Frobenius.

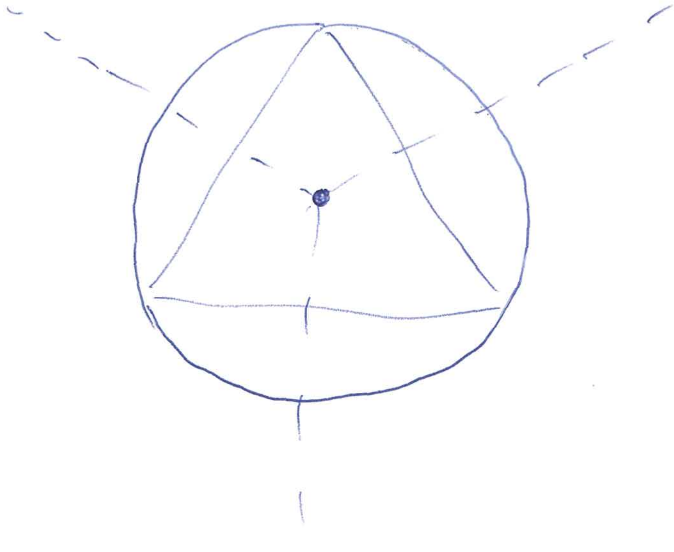
Exercise: Prove this.

~~Hint: First ignore convergence issues.~~

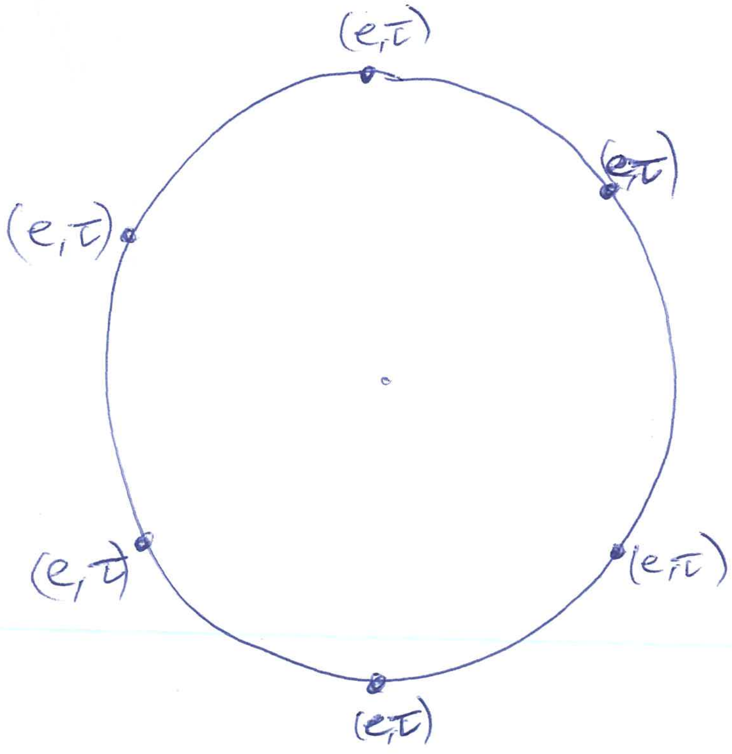
~~Hint: Reduce to a one dimensional~~

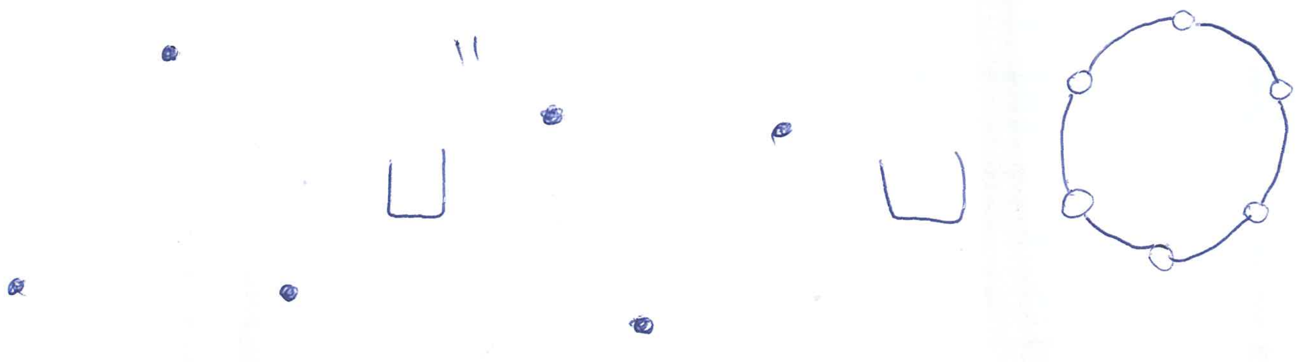
Hint: Interpret $F^n g^{-1}$ as a Frobenius and use the trace formula.

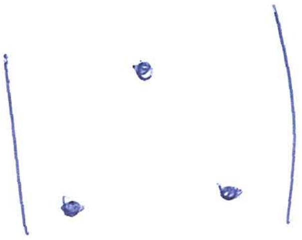
$S_3 \circlearrowright \mathbb{R}^2$ reflection rep.

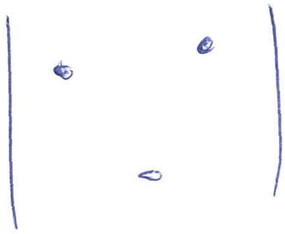


$S_3 \circlearrowright$ circle in here.

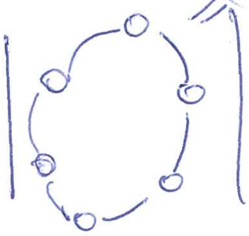
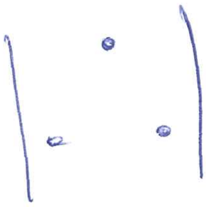




 $= I_{C_2}^{S_3}\mathbb{1} = \mathbb{1} \oplus V$

 $= I_{C_2}^{S_3}\mathbb{1} = \mathbb{1} \oplus V$

$\left(I_e^{S_3}\mathbb{1}\right)$ $\shortparallel$

 $-$  $-$  $+ \mathbb{1}$

is norm 1, orthogonal to our others, so we have constructed the final irrep of S_3 (sign).