Flag Varieties & Deligne-Lusztig Varieties

Sam Jeralds

Deligne-Lusztig Seminar

March 2025

§1. Recollections on flag varieties

Fix an algebraically-closed field $\mathbb{k}$, and G a connected reductive group $/\mathbb{k}$.
Recall that a Borel subgroup $B \leq G$ is a maximal closed connected solvable subgroup of G.

Our first goal is to understand the flag variety associated to G.
We motivate the definition by considering the classical case of GL_n.

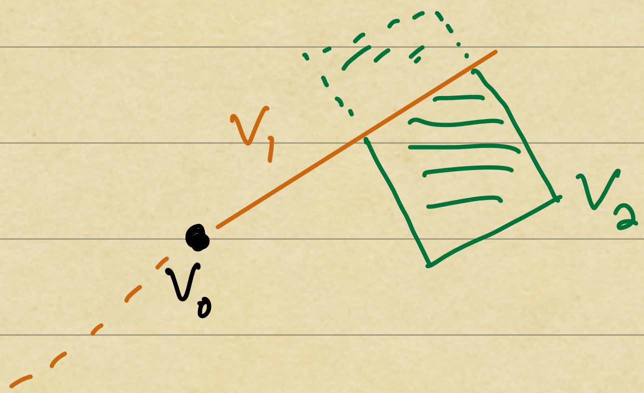
1. A: Flags in $\Bbbk^n$

• **Definition:** A flag in the vector space $\Bbbk^n$ is an increasing sequence of subspaces

$$\{0\} = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_n = \Bbbk^n$$

where $\dim V_i = i$

• **Example:** We imagine, in $\Bbbk^3$, building a "flag" on a "flagpole", via



• **Definition:** The (full/complete) flag variety is, as a set,

$$Fl_n := \{ (V_\bullet) : V_\bullet \text{ a flag in } \Bbbk^n \}$$

Note that $GL_n(\mathbb{k})$ acts on Fl_n by left multiplication, via compatible change of basis. Fix the "base-point flag"

$$V_\bullet^\circ := \{0\} \subset \langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \cdots \subset \langle e_1, \dots, e_n \rangle = \mathbb{k}^n$$

where $\langle\ \rangle$ denotes $\mathbb{k}$-span, and e_i the i^{th} standard basis element.

- Exercise: $Fl_n = GL_n(\mathbb{k}) \cdot (V_\bullet^\circ)$
- Exercise: $\mathrm{stab}_{GL_n(\mathbb{k})}(V_\bullet^\circ) = B_n = \left\{ \begin{pmatrix} * & \cdots & * \\ 0 & \ddots & * \end{pmatrix} \right\}$,

the standard Borel in $GL_n(\mathbb{k})$.

Together, these exercises allow us to give a geometry to Fl_n, via orbit-stabilizer.

- Proposition 1: $Fl_n \cong GL_n(\mathbb{k}) / B_n$

1. B: Structure Theory of Flag Varieties

Motivated by Proposition 1, we make the following

• **Definition:** Let G be a connected reductive group $/\mathbb{k}$, and $B < G$ a Borel subgroup. Then the flag variety associated to G is the homogeneous space $\mathcal{B} := G/B$

This formulation of the flag variety is beneficial in that it shows $\mathcal{B}$ is <u>smooth</u> (since it is homogeneous). But, it is less beneficial in that it seemingly depends on a choice of Borel. But, it in fact doesn't.

• **Proposition 2:**
$$G/B \xrightarrow{\sim} \{\text{Borel subgroups } B' < G\}$$
$$gB \longmapsto gBg^{-1}$$

– **Remark:** This relies on all Borels being conjugate, and B being self-normalizing.

A similar common alternative description is given by the following: Let $\mathfrak{g} = \mathrm{Lie}\, G$, $\mathfrak{b} = \mathrm{Lie}\, B$. Then we have

$$G/B \xrightarrow{\sim} \{\text{Borel subalgebras } \mathfrak{b}' \subset \mathfrak{g}\}$$
$$gB \longmapsto \mathrm{Ad}_g(\mathfrak{b})$$

But,

$\{\text{Borel subalgebras } \mathfrak{b}' \subset \mathfrak{g}\} \subseteq Gr(\dim \mathfrak{b}, \mathfrak{g})$, the Grassmannian of $\dim \mathfrak{b}$- subspaces of $\mathfrak{g}$.

• Exercise: The property of being a Borel subalgebra is a closed condition.

$\Rightarrow$ <u>$\mathcal{B}$ is projective</u>

Recall for G and a fixed B, we had the Bruhat decomposition

$$G = \bigsqcup_{w \in W} B \dot{w} B$$

for $w \in W$, the Weyl group

This gives the corresponding Bruhat decomposition on $\mathcal{B} = G/B$, via

$$G/B = \bigsqcup_{w \in W} B\dot{w}B/B$$

• **Definition:** We call the strata $B\dot{w}B/B$ a *Schubert cell* and its closure $\overline{B\dot{w}B/B}$ a *Schubert variety.*

In particular, the Bruhat decomposition gives a decomposition of $\mathcal{B}$ into B-orbits, with distinguished orbit representatives

$$\dot{w}B/B.$$

Each stratum $B\dot{w}B/B$ is affine, and in fact

$$B\dot{w}B/B \cong \mathbb{A}_{\mathbb{K}}^{l(w)}$$

where $l(w)$ is the length of w in the Weyl group.

1.C: Double Flag Varieties

- **Definition:** For G, B, and $\mathcal{B}$ as before, the double flag variety associated to G is given by
$$\mathcal{B} \times \mathcal{B} \cong G/B \times G/B$$

- **Exercise:** For any group G and subgroup H, there is a canonical bijection
$$\{H\text{-orbits on } G/H\} \longleftrightarrow \{G\text{-orbits on } G/H \times G/H\}$$
(diagonal action).

Particularly, we have canonical bijections
$$W \longleftrightarrow \{B\text{-orbits on } G/B\}$$
$$\updownarrow$$
$$\{G\text{-orbits on } G/B \times G/B\}.$$

- **Definition:** We denote by $\mathcal{O}(w)$ the G-orbit in $\mathcal{B} \times \mathcal{B}$ labelled by $w \in W$.

Concretely, for a fixed Borel as above, we have the constructions

$$\mathcal{O}(w) = G \times_B \left(B\dot{w}B/B \right)$$

$$\cong G.\left(eB/B,\ wB/B \right) \subset G/B \times G/B$$
$$= \{ (g_1 B, g_2 B) \mid g_1^{-1} g_2 \in B\dot{w}B \}$$

By the first construction, it's clear that
$$\dim \mathcal{O}(w) = \dim \mathcal{B} + \ell(w)$$

From the second, it's clear that
$$\mathcal{O}(1) = \Delta \mathcal{B} \subset \mathcal{B} \times \mathcal{B}, \quad \mathcal{O}(w_0) \subset \mathcal{B} \times \mathcal{B} \text{ dense}$$

- **Definition:** For two Borels $B_1, B_2 < G$, we say B_1 is in *relative position* w to B_2 if
$$(B_1, B_2) \in \mathcal{O}(w) \subset \mathcal{B} \times \mathcal{B}.$$
We denote this by
$$B_1 \xrightarrow{w} B_2.$$

§2. Deligne-Lusztig Varieties

We are now ready to discuss Deligne-Lusztig varieties. Take as before G connected reductive, and let $\mathbb{k} = \overline{\mathbb{F}_q}$. Assume G is defined over $\mathbb{F}_q$, and let F be the corresponding Frobenius map.

2.A: Deligne-Lusztig Varieties $X(w)$

Set $\Gamma_F = \{(B, F(B)) \mid B \in \mathcal{B}\}$, the graph of the Frobenius.

• **Definition:** For $w \in W$, the Deligne-Lusztig variety $X(w)$ is given by

$$X(w) = \mathcal{O}(w) \cap \Gamma_F$$
$$= \{ B \in \mathcal{B} \mid B \xrightarrow{w} F(B) \}$$

• **Remark:** By transversality of the intersection, $X(w)$ is smooth of pure dimension $\ell(w)$.

• Example: $X(e)$ is just the set of rational Borel subgroups, so

$$X(e) = \mathcal{B}^F$$
$$= G^F/B^F, \text{ if } B \text{ is F-stable.}$$

This is a dimension 0 subset in $\mathcal{B}$.

In general, we have that

$$\mathcal{B} = \bigsqcup_{w} X(w).$$

• Fix an F-stable Borel B, and recall that

$$G/B \xrightarrow{\sim} \mathcal{B}$$
$$gB \longmapsto gBg^{-1}$$

Using this Borel to define the base point, tracing through the associations we in fact get

$$X(w) \cong \{ gB \in G/B \mid g^{-1}F(g) \in B\dot{w}B \}$$

• Corollary: $G^F \curvearrowright X(w)$ by left multiplication.

2.B: Torsors over $X(w)$

Keeping the F-stable Borel B as above, let U be the unipotent radical of B.
Then we get a T-torsor over $\mathcal{B}$ given by

$$Y := G/U \twoheadrightarrow G/B = \mathcal{B}$$

Note that naturally T normalizes U, so T acts on Y from the right.

• Definition: For $w \in W$, define the locally-closed subvariety

$$Y(w) := \{ gU \in G/U \mid g^{-1}F(g) \in U\dot{w}U \}$$

This again has a natural left G^F action.
What about the right T-action?

For $t \in T$, $gU \in Y(w)$, we see that

$$(gt)^{-1} F(gt) = t^{-1} [g^{-1} F(g)] F(t) \in t^{-1} U \dot{w} U F(t)$$

Then

$$\begin{aligned} t^{-1} U \dot{w} U F(t) &= U t^{-1} \dot{w} U F(t) \\ &= U \dot{w} (\dot{w}^{-1} t^{-1} \dot{w}) U F(t) \\ &= U \dot{w} U (\dot{w}^{-1} t^{-1} \dot{w}) F(t) \end{aligned}$$

Thus $gtU \in Y(w)$ if and only if

$$\dot{w}^{-1} t^{-1} \dot{w} F(t) = 1$$

$$\begin{aligned} &\iff \dot{w} F(t) \dot{w}^{-1} = t \\ &\iff t \in T^{wF} \end{aligned}$$

where $\dot{w}F : T \to T$, $t \mapsto \dot{w} F(t) \dot{w}^{-1}$ is a Frobenius map for T.

- **Corollary:** $Y(w)$ has a right action of T^{wF}, which commutes with the left G^F action.

• Example: Let $G = SL_2$, $T \subset B \subset SL_2$ the standard diagonal and upper triangular Borel. Then $W = \{1, s\}$ where we pick representative

$$s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Of course, $\mathcal{B} = {}^{SL_2}/_B$

$$= \{ \{0\} \subset L \subset \mathbb{k}^2 \}$$

$$\cong \mathbb{P}^1$$

In $\mathcal{B} \times \mathcal{B}$ we have the two orbits

$$\mathcal{O}(1) \cong \Delta \mathbb{P}^1 \subset \mathbb{P}^1 \times \mathbb{P}^1$$

$$\mathcal{O}(s) \cong (\mathbb{P}^1 \times \mathbb{P}^1) \setminus \Delta \mathbb{P}^1$$

Thus, two flags $\{0\} \subset L \subset \mathbb{k}^2$, $\{0\} \subset L' \subset \mathbb{k}^2$ are in relative position s if and only if

$$L + L' = \mathbb{k}^2.$$

Next, $X(1) \cong \mathbb{P}^1(\mathbb{F}_q)$, $X(s) \cong \mathbb{P}^1 \setminus \mathbb{P}^1(\mathbb{F}_q)$.

A point in $Y = {}^{G}/_{U}$ over the flag $\{0\} \subset L \subset \mathbb{k}^2$ is the data of the flag along with choices of vectors $v \in L$, $u \in {}^{\mathbb{k}^2}/_{L}$ such that

$$\det(v, u) = 1.$$

- Remark: Note that s interchanges the roles of v and u. Thus...

A point in $Y(s)$ then corresponds to a flag $\{0\} \subset L \subset \mathbb{k}^2$ with a vector $v \in L$ such that

$$\det(v, F(v)) = 1$$

That is, if $v = \begin{bmatrix} x \\ y \end{bmatrix}$, then we need

$$\boxed{xy^q - x^q y = 1}$$

- Exercise: Check all of this example carefully!

- Exercise: Let $G = GL(V)$, $V = \mathbb{k}^n$, T the diagonal torus, and $\mathcal{B} = {}^{GL}/_{B}$ the full flag variety as in §1. Let $w = (1\,2\,3 \dots n)$ the n-cycle.

(a) Show two flags $V_\bullet$, $V_\bullet'$ satisfy relative position $V_\bullet \xrightarrow{w} V_\bullet'$ if and only if

$$V_i + V_i' = V_{i+1}$$

(b) $V_\bullet \xrightarrow{w} F(V_\bullet)$ iff $V_i = V_1 + F(V_1) + \dots + F^{i-1}(V_1)$

(c) Deduce that

$$Y(w) \cong \left\{ (x_1, \ldots, x_n) \in \mathbb{A}^n \,\middle|\, \det\left((x_i)^{q^{j-1}} \right)^{q-1} = (-1)^{n-1} \right\}$$