Exercises for Char. Sheaves

1. Let $T = \mathbb{G}_m$. Let $\mathcal{L}$ be the "$(q-1)^{th}$-root sheaf" on T, so that $\mathcal{L}$ is invariant under Fr^*, where $Fr: T \to T$ is $x \mapsto x^q$.
Let $\phi: Fr^*\mathcal{L} \xrightarrow{\sim} \mathcal{L}$ be the corresponding isomorphism.
Compute $\chi_{\mathcal{L},\phi}(x) := \sum_i (-1)^i \mathrm{Tr}(\phi \mid H^i_x(\mathcal{L}))$
for $x \in T(\mathbb{F}_q)$. (Note that $\mathcal{L}$ is just a sheaf, i.e. a complex supported in deg 0, so $H^0_x(\mathcal{L}) = \mathcal{L}_x$ and $H^i_x(\mathcal{L}) = 0$ oth.).

2. Here we compute the character sheaves for $G = PGL_2$. Recall our diagram

$$\begin{array}{ccc} \tilde{G} & \xrightarrow{h} & B/[B,B] \simeq T \\ \downarrow \pi & & \\ G & & \end{array}$$

where $\tilde{G} = \{(x, B) \in G \times \mathcal{B} \mid x \in B\}$.

a) Compute the fiber of π above
   i) A regular semi-simple elt of G
   ii) A nilpotent element
   iii) $id \in G$

b) Compute $\chi_{K_\mathcal{L}}$ for $K_\mathcal{L} := \pi_* h^* \mathcal{L}$, when $\mathcal{L}$ is
   i) the constant sheaf on T
   ii) the $(q-1)^{th}$-root sheaf