ETALE COHOMOLOGY EXERCISES

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The following exercises are not in order of difficulty, but starred exercises are harder.

Question 0.1. One may compute the (topological) cohomology of smooth projective algebraic varieties by counting points over finite fields. Count the points for some of the following varieties, and deduce the structure of their cohomology over the complex numbers:

(1) Affine space $\mathbb{A}^n$.
(2) Projective space $\mathbb{P}^n$.
(3) The Grassmannian of lines in three dimensional projective space, $G(2,4)$.
(4) The flag manifold for a (split) reductive group.
(5) (*) The elliptic curve $V(X^3+Y^3+Z^3)$ over the finite field $\mathbb{F}_7$ with seven elements.
(6) (**) The Picard variety $Pic(C)$ of a smooth projective curve C defined over a finite field.

Question 0.2. Give an example of a manifold with automorphism ϕ such that the (total) trace of ϕ on the compactly supported cohomology does not equal the number of fixed points. Why is this example different to the Frobenius automorphism? Can you find an automorphism of a manifold that does resemble the characteristic p behaviour of the Frobenius?

Question 0.3. Let M be a compact manifold with free action of a finite group G, with quotient M/G. Prove that the natural map

$$H^*(M/G, \mathbb{Q}) \to H^*(M, \mathbb{Q})^G$$

is an isomorphism. Feel free to assume that M/G comes equipped with a cell structure or triangulation.

Question 0.4. Let $V = \bigoplus_i V_i$ be a finite dimensional complex graded vector space with endomorphism F, such that the generalised eigenvalues of F on V_i have absolute value $q^{i/2}$. Come up with a way to compute the dimension of each of each V_i from the data of the supertraces of powers of F:

$$n \mapsto T(n) := \sum_i (-1)^i Tr(F^n, V_i)$$

Question 0.5. (*)Prove the formula for the Lefchetz number, the supertrace of a finite order element g on the etale cohomology of $X/\mathbb{F}_q$ is given by the limit as $t \mapsto \infty$ of the rational function

$$R(t) := -\sum_{i=1}^{\infty} |X^{F^n \circ g^{-1}}| t^n$$

You may assume g commutes with the Frobenius (though this isn't needed).