Finite groups of Lie type - Exercises

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Exercise 1. Let p be a prime number, and let $\mathbb{F}_p$ denote the field with p elements. Let F be a finite extension of $\mathbb{F}_p$. Show that for any subfield F' of F, there exists some $m \in \mathbb{N}$ such that F' is the set of fixed points of the p^m-Frobenius

$$\begin{aligned} \mathrm{Frob}_m : F &\to F \\ x &\mapsto x^{p^m}. \end{aligned}$$

Use this to show that if $(X, \mathcal{O}_X)$ is an affine algebraic subvariety of k^n such that $\mathcal{O}_X$ is generated by polynomials in $\mathbb{F}_{p^m}[x_1, \dots, x_n]$, then Frob_m maps X to X.

Exercise 2 (Jordan decomposition)**.** Let $q = p^k$, $G = \mathrm{GL}_n(\mathbb{F}_q)$ and $A \in G$.

1. Show that A is semisimple if and only if the order of A is coprime to p.
2. Show that A is unipotent if and only if the order of A is a power of p.

Exercise 3. Let W be the group D_{12}, the plane symmetries of the regular hexagon. Give two different Coxeter systems $(W, S_1), (W, S_2)$ for W with $|S_1| \neq |S_2|$.

Exercise 4. Let G be a connected reductive group over k. Prove the sharp Bruhat decomposition from Corollary 3.8. (Hint: It may be helpful to determine which root subgroups get normalised by $\dot{w}$).

Exercise 5. Use the sharp Bruhat decomposition for finite algebraic groups to count points of GL_n.

Exercise 6. Let $G = \mathrm{GL}_n(k)$ and let $T \subseteq G$ be the subgroup of diagonal matrices. Show that $N_G(T)$ is given by the subgroup of generalised permutation matrices (i.e. those matrices which have exactly one nonzero entry in each row and column). Do the same for $G = Sp_4$.

Exercise 7. Let $G = \mathrm{GL}_n(k)$ and let Q be the antidiagonal $(n \times n)$-matrix with antidiagonal entries $(1, \dots, 1)$. Let F_q be the Frobenius for the $\mathbb{F}_q$-rational structure on $\mathrm{GL}_n(k)$ and define the map

$$\begin{aligned} \gamma : \mathrm{GL}_n(k) &\to \mathrm{GL}_n(k) \\ M &\mapsto Q^{-1}(M^T)^{-1}Q. \end{aligned}$$

1. Verify that γ is a regular automorphism of finite order.

2. By Theorem 5.4, the map $F' := F_q \circ \gamma$ is again a Frobenius map for an $\mathbb{F}_q$-rational structure on $\mathrm{GL}_n(k)$. Determine the F'-fixed points of $\mathrm{GL}_n(k)$.

3. Justify why $\mathrm{GL}_n(k)^{F'}$ deserves to be called the unitary group $\mathrm{U}_n(\mathbb{F}_q)$. (Hint: It may be useful to draw an analogy with $U_n(\mathbb{C})$ as a subgroup of $\mathrm{GL}_n(\mathbb{C})$.)

Exercise 8. Prove Theorem 5.12. (Hint: To show that F preserves U, use abstract properties of the Jordan decomposition. To show part 2, use the Lang-Steinberg theorem.)

Exercise 9. Let G be a connected algebraic group acting transitively on a nonempty set X. Let $F : G \to G$ be a generalised Frobenius map and let $F' : X \to X$ be an automorphism of sets which is compatible with F in the sense that $F'(g.x) = F(g).F'(x)$ for all $g \in G$ and $x \in X$. Show that X has a F'-fixed point.

Exercise 10. Let $G = \mathrm{GL}_n$. Verify the details of Example 5.20 and determine the G^F-conjugation orbits of maximal tori in G. (Can you make the bijection explicit?)