1. Introduction to Deligne-Lusztig theory - Charlotte Chan

All representations are over $\mathbb{C}$.

For a finite group G, $|G| = \sum_{\pi \in Irr(G)} (\dim \pi)^2$

Reps of $GL_2\mathbb{F}_q = G$

- 1-dim. reps of $GL_2\mathbb{F}_q$: $GL_2\mathbb{F}_q \xrightarrow{\det} \mathbb{F}_q^\times \xrightarrow{\theta_0} \mathbb{C}$ ($q-1$ reps)
- $T = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \cong \mathbb{F}_q^\times \times \mathbb{F}_q^\times$. Surjection $B \to T$ gives B-rep
$$\begin{tikzcd} B \arrow[r, two heads] \arrow[rr, bend right, "\tilde\theta"'] & T \arrow[r, "\theta"] & \mathbb{C} \end{tikzcd}$$
$B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$

Induce to $GL_2\mathbb{F}_q$-rep $\mathrm{Ind}_B^G(\tilde\theta) = \mathrm{pInd}_T^G(\theta)$

<u>Exercise:</u>
$$\dim \mathrm{Hom}_G\left(\mathrm{Ind}_{B'}^G(\tilde\theta), \mathrm{Ind}_{B''}^G(\tilde\theta')\right) = \begin{cases} 2 & \text{if } \theta = \theta' = (\theta')^w \\ 1 & \text{if } \theta \in \{\theta', \theta'^w\} \text{ and } \theta' \neq (\theta')^w \\ 0 & \text{if } \theta \notin \{\theta', (\theta')^w\} \end{cases}$$
(B', B'' can be $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ or $\begin{pmatrix} * & 0 \\ * & * \end{pmatrix}$)

where $\theta^w\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} = \theta\begin{pmatrix} y & 0 \\ 0 & x \end{pmatrix}$

$$= \#\{w' \in \{1, w\} \text{ s.t. } \theta = (\theta')^w\}$$
$$= \sum_{w \in W} \dim \mathrm{Hom}_T(\theta, (\theta')^w)$$

In particular, $\mathrm{pInd}_T^G(\theta)$ does not depend on the choice of B,

and $\dim \mathrm{End}_G(\mathrm{pInd}_T^G(\theta)) = \begin{cases} 2 & \theta = \theta^w \\ 1 & \theta \neq \theta^w \end{cases}$

There are $(q-1)^2$ characters of $\mathbb{F}_q^\times \times \mathbb{F}_q^\times$ and $q-1$ of these have $\theta = \theta^w$,

so there are $\frac{(q-1)^2 - (q-1)}{2} = \frac{q^2-3q+2}{2}$ irreps of dimension $\frac{|G|}{|B|} = q+1$

<u>Exercise</u>: $\mathrm{pInd}_T^G(\theta \otimes (\theta_0 \circ \det)) = \mathrm{pInd}_T^G(\theta) \otimes (\theta_0 \circ \det)$,

hence if $\theta' = (\theta')^{w_0}$ then $\theta' = \theta_0 \circ \det$, then $\mathrm{pInd}_T^G(\theta') = \mathrm{pInd}_T^G(\mathbb{1}) \otimes (\theta_0 \circ \det)$

So decomposing $\mathrm{pInd}_T^G(\theta')$ is "the same as" decomposing $\mathrm{pInd}_T^G(\mathbb{1})$

$p\,\mathrm{Ind}_T^G(\mathbb{1}) = \{f : G \to \mathbb{C} \mid f(gb) = f(g) \ \forall g \in G, b \in B\} = \{f : G/B \to \mathbb{C}\}$

We have a copy of the trivial rep $\{\text{constant functions } G/B \to \mathbb{C}\} \subset p\,\mathrm{Ind}_T^G(\mathbb{1})$

We then have $p\,\mathrm{Ind}_T^G(\mathbb{1}) = \mathbb{1} \oplus \mathrm{St}_G$ ← Steinberg representation, dim q

⇒ We have $q-1$ irreps $\mathrm{St}_G \otimes (\theta_0 \circ \det)$ of dimension q.

Count the irreps and check the dimension formula works

$$(q-1)(1)^2 + (q-1)(q)^2 + \frac{q^2-3q+2}{2}(q+1)^2 \leq (q^2-1)(q^2-q) = |G|$$

biggest terms in each are $\frac{q^4}{2}$ and q^4, so we're missing half the irreps.

Where are the others? Answer comes from Deligne-Lusztig theory!

Suppose $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2\mathbb{F}_q$ has distinct eigenvalues. These eigenvalues are either in $\mathbb{F}_q^\times$ or $\mathbb{F}_{q^2}^\times$ (since characteristic polynomial has deg 2).

<u>Question</u>: what about tori with elements whose eigenvalues are not in $\mathbb{F}_q^\times$?

(a $\overset{\text{maximal}}{\vee}$ torus in GL_2 is a subgroup scheme T s.t. $T_{\overline{\mathbb{F}_q}} \cong (\mathbb{G}_m)^2$)

Analogue: $GL_2\mathbb{R}$ has a non-split torus $\left\{\begin{pmatrix} r\cos\theta & r\sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix} \middle| r, \theta \in \mathbb{R}\right\}$

$$\begin{tikzcd}
\mathbb{C}^\times \arrow[r] \arrow[d, phantom, "\shortparallel"] & GL_2\mathbb{R} \arrow[dd, phantom, "\cup"] \\
\mathbb{R}[\sqrt{-1}]^* \arrow[d, phantom, "\cup"] & \\
r\cos\theta + ir\sin\theta \arrow[r, mapsto] & \begin{pmatrix} r\cos\theta & r\sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix}
\end{tikzcd}$$

Similarly, we consider a number d without a square root in $\mathbb{F}_q$, so

$$\begin{tikzcd}
\mathbb{F}_{q^2}^\times \arrow[r] \arrow[d, phantom, "\shortparallel"] & GL_2\mathbb{F}_q \\
(\mathbb{F}_q[\sqrt{d}])^\times \arrow[d, phantom, "\cup"] & \\
a+b\sqrt{d} \arrow[r, mapsto] & \begin{pmatrix} a & b \\ bd & a \end{pmatrix} \arrow[uu]
\end{tikzcd}$$

(Exercise: what if $q = 2^n$?)

No Borel (maximal solvable connected subgroup) contains this nonsplit torus.

DL induction gives a way to associate to any $\theta : \mathbb{F}_{q^2}^\times \to \mathbb{C}^\times$ a G-character $R_T^G(\theta)$ which behaves like parabolic induction.

Theorem. Let T, T' be a maximal tori of $GL_2\mathbb{F}_q$ and θ, θ' be characters of T, T' respectively. Then

${}^w\theta'(t) = \theta'(gtg^{-1})$ for $g \in w$

$$\langle R_T^G(\theta), R_T^G(\theta') \rangle = \#\{ w \in W_{T,T'} \mid \theta = {}^w\theta' \}$$

where $W_{T,T'} = \{ g \in GL_2\mathbb{F}_q \mid gTg^{-1} = T' \}/T$ = Weyl group if $T = T'$

Example:
- $T \cong \mathbb{F}_q^\times \times \mathbb{F}_q^\times$, $W_T = S_2 = \{1, w_0\} \to w_0(a,b) = (b,a)$
- $T \cong \mathbb{F}_{q^2}^\times$, $W_T = S_2 = \{1, w_0\} \to w_0(a) = a^q$
  OR $w_0(a_0 + b_0\sqrt{d}) = a_0 - b_0\sqrt{d}$

Lemma: $R_T^G(\theta \otimes (\theta_0 \circ \det)) = R_T^G(\theta) \otimes (\theta_0 \circ \det)$

Fact: $R_T^G(\theta) = -H_c^1(\text{some curve})_\theta + H_c^2(\text{some curve})_\theta$

so $R_T^G(\theta)$ might only be a virtual character since it may be negative.

Example: $R_T^G(\mathbb{1}) = -St_G + 1$. Note: $\langle 1 + St, 1 - St \rangle = 1 - 1 = 0$

Upshot is, we have more irreps now of dimension $(q-1)$:

$$(q-1)(1)^2 + (q-1)q^2 + \left(\frac{q^2-3q+2}{2}\right)(q+1)^2 + \left(\frac{q^2-3q+2}{2}\right)(q-1)^2 \overset{\text{check}}{=} |G|$$

Theorem (DL): If ρ is an irreducible representation of a finite group of Lie type G, then there is a maximal torus T and character $\theta : T \to \mathbb{C}^\times$ s.t. $\langle \rho, R_T^G(\theta) \rangle \neq 0$.

What is the curve used to define $R_T^G(\theta)$? Basic framework: we want a bijective map $\{(T,\theta)\}/_W \longrightarrow Irr(G)$. That such a map might exist was observed after hand computations for GL_2, "Macdonald's conjecture". DL theory uses T to construct a variety $X_{T\subset G}$ which "almost" lives inside the flag variety G/B. $X_{T\subset G}$ has natural actions by T on the left and G on the right, and then the cohomology modules give a G-character

$$R_T^G(\theta) := \sum (-1)^i H_c^i(X_{T\subset G})_\theta$$

where θ subscript denotes the θ-isotypic subspace. For $G = GL_2\mathbb{F}_q$ with T the nonsplit torus, we have

$$X_{T\subset G} = \mathbb{V}\left((a^q b - b^q a)^{q-1} = 1\right) \subset \mathbb{A}^2_{\overline{\mathbb{F}_q}}$$

$\nwarrow$ "Drinfeld curve"

- $GL_2\mathbb{F}_q$ acts on $\binom{a}{b}$ by matrix mult.
- $\mathbb{F}_{q^2}^\times$ acts on $\binom{a}{b}$ by scaling.

Def (DL variety): Let G be a connected alg. group over $\overline{\mathbb{F}_q}$ and $F: G \to G$ the Frobenius map for a $\mathbb{F}_q$-rational structure of G, and $T \subset B \subset G$ maximal torus and Borel with T stable under F.

$$X_{T\subset G} := \{g \in G \mid g^{-1}F(g) \in U\}$$

where $U = R_u(B)$. (Note: this depends on B, but R_T^G doesn't).

__Exmple__: $GL_2(\overline{\mathbb{F}}_q) \xrightarrow{F} GL_2(\overline{\mathbb{F}}_q)$

$$g \longmapsto w_0^{-1} g^q w_0$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto w_0^{-1} \begin{pmatrix} a^q & b^q \\ c^q & b^q \end{pmatrix} w_0$$

① $GL_2(\overline{\mathbb{F}}_q)^F \cong GL_2(\mathbb{F}_q)$

② $T = \{\begin{pmatrix} * & \\ & x \end{pmatrix}\} \subset GL_2 \overline{\mathbb{F}}_q$ gives $T^F = \{\begin{pmatrix} a & \\ & b \end{pmatrix} \mid a = b^q, b = a^q\} \cong \mathbb{F}_{q^2}^\times$

With $B = \begin{pmatrix} * & * \\ & x \end{pmatrix}$, $U = \begin{pmatrix} 1 & * \\ & 1 \end{pmatrix}$, we have

$$\begin{aligned} X_{T\subset G} &= \{g \in GL_2(\overline{\mathbb{F}}_q) \mid g^{-1} F(g) \in U\} \\ &= \{g \mid F(g) \in gU\} \\ &= \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid \begin{pmatrix} d^q & c^q \\ b^q & a^q \end{pmatrix} = \begin{pmatrix} a & ax+b \\ c & cx+d \end{pmatrix} \text{ for some } x \in \overline{\mathbb{F}}_q\} \\ &= \{\begin{pmatrix} a & b^q \\ b & a^q \end{pmatrix} \mid \det \in \mathbb{F}_q^\times\} \\ &\cong \mathbb{V}((a^{q+1} - b^{q+1})^{q-1} = 1) \\ &\cong \mathbb{V}((a^q b - b^q a)^{q-1} = 1) \quad \text{(exercise)} \end{aligned}$$

__Fact__: the Drinfeld curve has image under $\mathbb{A}^2 \longrightarrow \mathbb{P}^1(\mathbb{F}_q)$ given by $\mathbb{P}^1 \setminus \mathbb{P}^1(\mathbb{F}_q)$, and all fibres are iso. to $\mathbb{F}_{q^2}^\times$

__$GL_2 \mathbb{F}_q$ vs $SL_2 \mathbb{F}_q$__. Recall $\langle p\mathrm{Ind}_T^G(\theta), p\mathrm{Ind}_T^G(\theta)\rangle = \#\{w \in W \mid \theta^w = \theta\}$.

There is exactly one W-orbit of θ such that $\theta \neq \theta^w$ & $\theta|_{SL_2 \cap T} = \theta^w|_{SL_2 \cap T}$

This also happens for the nonsplit torus. (Exercise)

$\Rightarrow$ There's one $p\mathrm{Ind}(\theta)$ and one $R^G_{T_{n.s.}}(\theta)$ which are irreducible for $GL_2 \mathbb{F}_q$ but not for $SL_2 \mathbb{F}_q$. Each splits into 2 nonisomorphic reps with dim $\frac{q-1}{2}$ for the former and $\frac{q+1}{2}$ for the latter.

There is a "spectrum" from split to elliptic maximal tori: