Deligne - Lusztig Induction

- Characters of $GL_2(\mathbb{F}_q)$, $SL_2(\mathbb{F}_q)$ have been known for some time, certainly to Frobenius.

- Green (1955) determined the character table of $GL_n(\mathbb{F}_q)$. Essentially by a lot of hard work.

- Srinivasan (1968) then worked out the character table of $Sp_4(\mathbb{F}_q)$.

Based on these known character tables this led Macdonald (1968) to conjecture there should be some well-defined correspondence between

(T, θ) and irreducible representations of G^F

T: F-stable torus; θ: character of T

(stating this imprecisely)

For $GL_2(\mathbb{F}_q)$: Since $B = U \rtimes T$, any $\lambda \in \mathrm{Hom}(T, \overline{\mathbb{Q}}_\ell^\times)$ extends to B. That is,

$$\lambda\begin{pmatrix} a & b \\ & d \end{pmatrix} = \lambda_1(a)\lambda_2(d).$$

Note: $\widehat{T} \simeq \widehat{\mathbb{F}_q^\times} \times \widehat{\mathbb{F}_q^\times}$. So we may induce!

$$\mathrm{Ind}_B^G(\lambda) \simeq \{ f : G \to \overline{\mathbb{Q}}_\ell \mid f(gb) = \lambda(b) f(g) \}$$

whenever inverses should go...

Looks a lot like
$\mathrm{Fun}(\mathbb{P}^1(\mathbb{F}_q), \overline{\mathbb{Q}}_\ell)$!

It's dimension is $|G/B| = q+1$.

How many? $\lambda_1 \neq \lambda_2$ so $(q-1)(q-2)$ many

but $\lambda\begin{pmatrix} a & \\ & d \end{pmatrix}$, $\lambda\begin{pmatrix} d & \\ & a \end{pmatrix}$ give isomorphic reps

so $\dfrac{(q-1)(q-2)}{2}$ many.

<u>Exercise</u>. Prove this.

Another $q-1$ many: Take $\lambda \in \mathrm{Hom}(\mathbb{F}_q^\times, \overline{\mathbb{Q}_\ell})$, then we have a one-dimensional ③

$$g \longmapsto \lambda(\det g)$$

Recall, $\mathrm{Ind}_B^G(\lambda_1, \lambda_2)$ is reducible when $\lambda_1 = \lambda_2$. This decomposes into the det rep. and then an irreducible complement. Show this! It has dimension q, and there are $q-1$ many.

What remains are the tricky ones. We know there are $\frac{1}{2} q(q-1)$ left to be found.

The non-split torus $(T_s)_{\mathbb{F}_{q^2}} \simeq \mathbb{G}_{m, \mathbb{F}_{q^2}}$

$\mathbb{F}_{q^2}$ a 2-dim v.s. / $\mathbb{F}_q$. Multiplication identifies $\mathrm{End}_{\mathbb{F}_q}(\mathbb{F}_{q^2}) \simeq GL_2(\mathbb{F}_q)$

$$GL_2(\mathbb{F}_q) \ni \begin{pmatrix} x & dy \\ y & x \end{pmatrix} \quad \text{with} \quad \underbrace{x + y\sqrt{d}}_{a} \in \mathbb{F}_{q^2}^\times$$

$d \in \mathbb{F}_q^\times \setminus (\mathbb{F}_q^\times)^2$

Note $\det\begin{pmatrix} x & dy \\ y & x \end{pmatrix} = \mathrm{Norm}_{\mathbb{F}_{q^2}/\mathbb{F}_q}(x+\sqrt{d}y) = a \cdot a^q = a^{q+1}$.

So for any character

$$\theta : \mathbb{F}_{q^2}^{\times} \longrightarrow \overline{\mathbb{Q}_\ell}^{\times}$$

we "induce" again. $\mathrm{Ind}_{\mathbb{F}_{q^2}}^{G} \theta$ has dim. $q^2 - 1$. Note $\mathrm{Ind}(\theta) \simeq \mathrm{Ind}(\theta^q)$.

We get $\frac{1}{2} q(q-1)$ distinct reps when $\theta \overset{?}{\neq} \theta^q$. <u>NOT</u> irreducible however...

See exercises.

Deligne - Lusztig Induction

Recall the definition Sam gave last week.

$$\begin{array}{ccc} G/U & & Y(w) = \{gU \in G/U : g^{-1}F(g) \in U\dot{w}U\} \\ \downarrow & & \Big\downarrow T^{wF}\text{- torsor} \\ G/B & & X(w) = \{gB \in G/B : g^{-1}F(g) \in B\dot{w}B\} \end{array}$$

Exercise: Get the Drinfeld curve. That is,

$$\underset{\substack{\| \\ SL_2}}{G} = B \sqcup B\dot{s}B \longrightarrow B\dot{s}B = \left\{\begin{pmatrix} * & * \\ x & * \end{pmatrix} : x \neq 0\right\}$$

Great definition, but one may take some issue with it. $Y(w) \to X(w)$ is a G^F-equivariant T^{wF}-torsor. A priori, it's not clear that this is the $\mathbb{F}_q$-points of some maximal torus. So one can make the 'definition'

$$\begin{array}{l} Y_T = \{g \in G : g^{-1}F(g) \in F(U)\}/(U \cap F(U)) \\ \downarrow \\ X_T = \{g \in G : g^{-1}F(g) \in F(U)\}/(T^F \cdot (U \cap F(U))), \end{array}$$

Essentially just $\mathcal{L}^{-1}(U)$.

which is a T^F-torsor!

or whatever

So $G^F \times T^F$ acts on Y : $(g,t)\cdot x = g x t^{-1}$

Let $\ell \neq p$ be a prime. Then

$$G^F \times T^F \curvearrowright H^i_c(Y, \overline{\mathbb{Q}}_\ell).$$

Let $\theta : T^F \longrightarrow \overline{\mathbb{Q}}_\ell^\times$ be a character.

<u>Defn</u> The Deligne-Lusztig reps are

$$R^G_T : \mathrm{Hom}(T^F, \overline{\mathbb{Q}}_\ell^\times) \longrightarrow K_0 \operatorname{Rep} G^F$$

$$\theta \longmapsto \sum_{i \geq 0} (-1)^i H^i_c(Y, \overline{\mathbb{Q}}_\ell)_\theta$$

<u>Prop</u>. If $T \subset B$ and B is F-stable, then

immediately one has

$$\begin{array}{c} (G/U)^F \simeq G^F/U^F \\ \big\downarrow \\ (G/B)^F \simeq G^F/B^F \end{array} \qquad H^i(G^F/U^F, \overline{\mathbb{Q}}_\ell) = \begin{cases} \overline{\mathbb{Q}}_\ell[G^F/U^F] \\ 0 \quad i \geq 1 \end{cases}$$

so $\{ f : G^F \to \overline{\mathbb{Q}}_\ell \mid f|_{U^F} = 1,\ f_{T^F} = \theta \}$

$= \operatorname{Ind}_{B^F}^{G^F} \theta$!

<u>Example</u>. $G = SL_2$ $\quad T$ split torus $\subset B$

T_s non-split torus $\simeq \mu_{q+1}$

As before $Y_T \simeq G^F/U^F$, so

$$R_T^G(\theta) = \mathrm{Ind}_{B^F}^{G^F}(\theta).$$

Y_{T_s} is the Drinfeld curve

$$Y = \{(x,y) \in \mathbb{A}^2 : xy^q - x^q y = 1\}$$

Action $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (x,y) = (ax+by, cx+dy)$

and $\mu_{q+1} \curvearrowright Y$ by $\lambda \cdot (x,y) = (\lambda x, \lambda y)$

since $\lambda x (\lambda y)^q - (\lambda x)^q y = \lambda^{q+1} xy^q - \lambda^{q+1} x^q y = 1$.

<u>The cohomology groups</u> $H^i(Y, \overline{\mathbb{Q}}_\ell)$

- $H_c^0(Y, \overline{\mathbb{Q}}_\ell) = 0$ by vanishing

$\boxed{\begin{array}{l} X \text{ affine + smooth} \\ H^i(X, \overline{\mathbb{Q}}_\ell) = 0 \\ 0 \le i < \dim X \end{array}}$

- $H_c^2(Y, \overline{\mathbb{Q}}_\ell)$ one-dimensional

(Y 1-dim and irreducible
trivial $G^F \times T^F$ action)

- $H_c^1(Y, \overline{\mathbb{Q}}_\ell)$ is the interesting "new" bit.

Since $G^F \times T^F \curvearrowright H^2(Y, \overline{\mathbb{Q}}_\ell)$ trivially, this means taking θ-isotypic part in ~~Pic~~ 0 unless $\theta = 1$.

When $\theta = 1_{\mu_{q+1}}$. Because of excision LES (8)

we have $H_c^*(Y)_{1_{\mu_{q+1}}} \simeq H^*(Y/\mu_{q+1}) \simeq H^*(X)$

so $H_c^*(Y)_{\theta 1_{\mu_{q+1}}} = H_c^*(\mathbb{P}^1) - H_c^*(\mathbb{P}^1(\mathbb{F}_q))$

Somethings we know:

$$H_c^i(\mathbb{P}^1) = \begin{cases} 1_{G^F} & i=0 \\ 0 & i=1 \\ 1_{G^F} & i=2 \end{cases}$$

$H^0(\mathbb{P}^1(\mathbb{F}_q))$

$= \overline{\mathbb{Q}}[G^F/B^F]$

$= \mathrm{Ind}_{B^F}^{G^F} 1 = 1 \oplus St$

so

$$(1_G - 0 + 1) - \mathrm{Ind}_{B^F}^{G^F} 1$$

$$= 2 \cdot 1_G - \mathrm{Ind}_{B^F}^{G^F} 1$$

$$= 2 \cdot 1 - 1 - St$$

$$= 1 - St \quad !$$

Now we want the case where θ is non-trivial

Recall trace formula (Chris)

$$\mathrm{trace}(z : H_c^*(Y)) = \mathrm{trace}(1, H_c^*(Y^z))$$

Recall,

$$\mathrm{trace}(g; H_c^*(Y)) = \mathrm{trace}(1, H_c^*(Y^g)^g).$$

If $z \in \mu_{q+1}$ not equal to 1, then

Y^z is empty! So $\mathrm{trace}(z, H_c^*(Y)) = 0$.

This means $H_c^*(Y)$ is a multiple of the regular rep. $\bar{\mathbb{Q}}_\ell[G^F/B^F]$. Each irrep of μ_{q+1} occurs in the same degree:

Degree of $H_c^*(Y)$ ~~occurs~~ is in same degree as $St - 1_G$

$\Rightarrow \dim = q-1$.

Or rather $-R_{T_S}^G(\theta)$ has deg. $q-1$.

After some Mackey stuff, we find it's irreducible for $\theta^2 \neq 1$.

Some facts:

- Every irreducible rep. of G^F occurs as an irreducible constituent of some $R_T^G \theta$; i.e. there is some (T,θ) s.t. $\langle \rho, R_T^G(\theta)\rangle \neq 0$.

- $R_T^G(\theta)$ is irreducible (up to a sign) when θ is in <u>general position</u>, that is $w\cdot\theta \neq \theta$ for any <u>non-trivial</u> $w \in W$.
  - Think of the $GL_2(\mathbb{F}_q)$ example: $\mathrm{Ind}_B^G(\lambda_1,\lambda_2)$ irreducible when $\lambda_1 \neq \lambda_2$ same as saying $s\cdot(\lambda_1,\lambda_2) = (\lambda_2,\lambda_1)$.

  - $SL_2(\mathbb{F}_q)$ example: $s\cdot\theta = \theta^{-1} \neq \theta \iff \theta^2 \neq 1$.

- We can put an inner product on $K_0 \operatorname{Rep} G^F$ like usual, but now there's extra subtlety involved.

- Scalar product formula. T, T' F-stable, and $\theta \in \operatorname{Irr} T^F$, $\theta' \in \operatorname{Irr} T'^F$. Then

$$\langle R_T^G(\theta), R_{T'}^G(\theta')\rangle = \frac{|\{g \in G^F : gTg^{-1} = T',\ g\theta g^{-1} = \theta'\}|}{|T^F|}$$

Bounds # of irreps in an $R_T^G(\theta)$ by $|W|$!

But we're working with virtual characters

<u>Prop</u> $R_T^G(\theta)$ and $R_{T'}^G(\theta')$ are either equal or orthogonal to eachother.

So it doesn't tell us terribly much about irreducible constituents, compared to the character theory of ordinary characters.

Eg: $\langle R_T^G 1, R_{T_s}^G 1 \rangle = \langle 1+St, 1-St \rangle = 0$.

Above tells us $R_T^G(\theta) = R_{T'}^G(\theta')$ iff there is some $g \in G^F$ s.t. $gTg^{-1} = T'$ and $g \cdot \theta = \theta'$.

To talk about the other extreme - where $R_T^G(\theta)$ and $R_{T'}^G(\theta')$ <u>share no</u> irreducible constituents. We need something stronger.

<u>Defn</u> Geometric conjugacy

(T, θ) and (T', θ') are geometrically conjugate if there is some $n \in \mathbb{N}$ and $g \in G^{F^n}$ s.t. $gTg^{-1} = T$ and

$g\theta \circ N_{F^n/F}\, g^{-1} = \theta' \circ N_{F^n/F}$.

Norm map $N_{F^n/F} : \mathbb{F}_{q^n} \to \mathbb{F}_q$, so $\theta \circ N : T^{F^n} \to \overline{\mathbb{Q}_\ell}^\times$

The idea now is to make this more 'geometric'. Namely, working in the group G, and the duality $X(T)$, $Y(T)$.

Fix $\overline{\mathbb{F}}_q^{\times} \xrightarrow{\sim} (\mathbb{Q}/\mathbb{Z})_{p'}$, $\qquad \varinjlim \mathbb{F}_{q^n}^{\times}$

$$\overline{\mathbb{F}}_q^{\times} \overset{\iota}{\hookrightarrow} \overline{\mathbb{Q}}_\ell^{\times}$$

$$\begin{array}{ccc} X(T) & \longrightarrow & \mathrm{Irr}(T^F) \\ \| & & \| \\ \mathrm{Hom}(T, \mathbb{G}_m) & & \mathrm{Hom}(T^F, \overline{\mathbb{Q}}_\ell^{\times}) \\ & & \\ \alpha & \longmapsto & \iota \circ \alpha|_{T^F} \end{array}$$

by restricting and identify $\overline{\mathbb{F}}_q^{\times} \hookrightarrow \overline{\mathbb{Q}}_\ell^{\times}$.

So

$$0 \longrightarrow X(T) \longrightarrow X(T) \longrightarrow \mathrm{Irr}\, T^F \longrightarrow 0$$

is exact.

Similarly

$$\begin{array}{ccc} Y & \longrightarrow & T^F \\ \| & & \\ \mathrm{Hom}(\mathbb{G}_m, T) & & \\ & & \\ y & \longmapsto & N_{F^n/F}\left(y\left(\frac{1}{q^n-1}\right)\right) \end{array}$$

This makes geometric conjugacy into the following:

(T, θ) and (T', θ') are geometrically conjugate by some $g \in G$ iff
(note: not G^F)

$$gTg^{-1} = T' \quad \text{and} \quad g\theta g^{-1} = \theta'$$

considered as belonging to

$$\mathrm{Hom}(Y(T), (\mathbb{Q}/\mathbb{Z})_{p'})$$

why?

A character of T^F $\longleftrightarrow$ character of $Y(T)$ w. values in p'-roots of unity in $\overline{\mathbb{Q}}_\ell^\times$.

<u>Dual groups</u> G in the data of

$$(X(T), Y(T), R, R^\vee)$$

G^* is its dual if there's an iso

$$X(T) \longrightarrow Y(T^*)$$

T^* maximal torus of G^*

inducing isomorphic root datum

$$(X(T), Y(T), R, R^\vee) \longleftrightarrow (Y(T^*), X(T^*), R^{*\vee}, R^*).$$

Everything we've just discussed implies

$$\mathrm{Irr}(T^F) = (T^*)^{F^*}$$

$\Longrightarrow$

$$\left\{ \begin{array}{c} \text{Geometric conjugacy} \\ \text{classes of pairs} \\ (T,\theta)\ (T',\theta') \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} F^*\text{-stable} \\ \text{semisimple } \cancel{\text{conj}} \\ \cancel{\text{classes}}\ \text{element} \\ \text{in } G^* \end{array} \right\}$$

$\swarrow\!\!\nearrow$ Intersection.

$$\left\{ \begin{array}{c} \text{semisimple conjugacy} \\ \text{classes in } G^{*F^*}. \end{array} \right\}$$

Can define the Lusztig series now. We have a surjective map

$$\begin{tikzcd} \mathrm{Irr}\, G^F \arrow[r] \arrow[dr] & \text{Pairs } (T,\theta) / \text{geo conj} \arrow[d, leftrightarrow] \\ & \text{semisimple classes in } G^{*F^*} \end{tikzcd}$$

well-defined by disjointness.

Defn

$$\mathcal{E}(G^F,(s)) = \left\{ \begin{array}{l} \text{Irreducibles of } G^F \text{ in} \\ \text{the DL rep associated to } (s) \end{array} \right\}$$

Basically just the fibre of the above.