Deligne-Lusztig theory exercises week 3

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1. Let $\mathbf{G} = \mathrm{GL}_n(\overline{\mathbb{F}}_p)$ and let $\mathbf{T}$ be the subgroup of diagonal matrices. Let F be the map raising matrix entries to the power of q, and let $F''(M) = F(J_n(M^\top)^{-1}J_n)$ where J_n is the matrix with 1s along the top-right to bottom-left diagonal.
   (a) Describe the elements of $\mathbf{T}^{F''}$, where $\mathbf{T} \subseteq \mathbf{G}$ is the subgroup of diagonal matrices.
   (b) Find the generators and relations for the Weyl group $\mathbf{W}^{F''}$ as a Coxeter group.
2. Prove that if an F-stable Levi subgroup $\mathbf{L} \subseteq \mathbf{G}$ is contained in two different F-stable parabolic subgroups $\mathbf{P}, \mathbf{Q}$, then the characters of $\mathbf{R}^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}}$ and $\mathbf{R}^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{Q}}$ are equal.
   Hint: consider $\langle \mathbf{R}^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}}\chi, \mathbf{R}^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{Q}}\chi\rangle$ for an arbitrary character χ of $\mathbf{L}$. See [DM, Theorem 5.3.1] for solution.
3. Prove that if $(\mathbf{L}, \Lambda)$ and $(\mathbf{L}', \Lambda')$ are cuspidal pairs for $\mathbf{G}^F$, then they have the same Harish-Chandra series if they are $\mathbf{G}^F$-conjugate, and otherwise have disjoint series.
   See [DM, Theorem 5.3.7] for solution.
4. [Bo, Exercise 3.2] For either $G = \mathrm{SL}_2(\mathbb{F}_q)$ or $G = \mathrm{GL}_2(\mathbb{F}_q)$, show that
   $$\mathrm{St}_G(g) = \begin{cases} |C_G(g)|_p & \text{if } g \text{ is semisimple} \\ 0 & \text{otherwise} \end{cases}$$
   where $C_G(g)$ is the centraliser $\{h \in G \mid hg = gh\}$ and $|\Gamma|_p$ is the largest power of p dividing $|\Gamma|$. *Hint: the matrices $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 0 \\ a & 1 \end{smallmatrix}\right)$ for $a \in \mathbb{F}_q$ are a complete set of coset representatives for $B = \{\left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)\}$.*
5. Let $\mathbf{G} = \mathrm{GL}_n(\overline{\mathbb{F}}_p)$, let $\mathbf{B}$ be the subgroup of upper-triangular matrices, and let $\mathbf{T}$ be the subgroup of diagonal matrices. Let F be the map raising matrix entries to the power of q, so $\mathbf{W}^F \cong \mathbf{W}$.
   (a) Describe the simple representations Λ of $\mathbf{T}^F$, and show that the relative Weyl group $\mathbf{W}_{\mathbf{G}}(\mathbf{T}, \Lambda)$ is a product of symmetric groups.
   (b) Show that the parameters $q_i = |\mathbf{B}^F|/|s_i\mathbf{B}^F s_i^{-1} \cap \mathbf{B}^F|$ in the Hecke algebra $\mathcal{H}(\mathbf{T}, \mathbb{1})$ are all equal to q.
   (c) Explain why $R^{\mathbf{G}}_{\mathbf{T}}\mathbb{1} = \mathbb{1} \oplus \mathrm{St}_{\mathbf{G}}$ when $n = 2$.
   (d) Find an isomorphism $\mathcal{H}(\mathbf{T}, \mathbb{1}) \cong \Bbbk\mathbf{W}$ when $n = 2$, and use this to find an explicit description of $\mathrm{St}_{\mathrm{GL}_2}$ (as a representation instead of as a character).

References

[Bo] C. Bonnafé: Representations of $\mathrm{SL}_2(\mathbb{F}_q)$. Springer, 2011.

[DM] F. Digne, J. Michel: Representations of finite groups of Lie type. Cambridge University Press, 1991.