Exercises (Introduction on Deligne-Lusztig Theory)

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Here are the exercises Charlotte presented during her lecture (see also the notes available on the seminar webpage taken by Joe Newton).

1. Let $\tilde{\Theta}, \tilde{\Theta}'$ be two characters of Borel subgroups B', B'' with common torus T in $G = GL_2(\mathbb{F}_q)$ such that $\tilde{\Theta}, \tilde{\Theta}'$ are obtained by characters Θ, Θ' of T, which are lifted to B' resp. B''. Show that the resulting characters when induced to G are invariant under B', B'', i.e. we have

$$\dim \mathrm{Hom}_G(\mathrm{Ind}_{B'}^G(\tilde{\Theta}), \mathrm{Ind}_{B''}^G(\tilde{\Theta})) = \#\{w \in W \mid \Theta =^w \Theta'\},$$

where W is the Weyl group of G (isomorphic to $S_2 = \{1, w_0\}$ in this case) and ${}^w\Theta'$ is the character given by ${}^w\Theta'(g) = \Theta'(w^{-1}gw)$. This justifies the notion pInd_T^G for parabolic induction, i.e. not mentioning the specific choice of B.

2. Show for a $\mathbb{F}_q^*$ character Θ_0,we have $\mathrm{pInd}_T^G(\Theta \otimes (\Theta_0 \circ \det)) = \mathrm{pInd}_T^G(\Theta) \otimes (\Theta_0 \circ \det)$, hence if $\Theta' = \Theta'^{w_0}$ then $\Theta' = \Theta_0 \circ \det$, then $\mathrm{pInd}_T^G(\Theta') = \mathrm{pInd}_T^G(1) \otimes (\Theta_0 \circ \det)$.So decomposing $\mathrm{pInd}_T^G(\Theta')$ is "the same as" decomposing $\mathrm{pInd}_T^G(1)$.

3. We can obtain a non-split torus in $GL_2(\mathbb{F}_q)$ for $q = p^n, p$ odd by the map

$$\mathbb{F}_{q^2}^* \to GL_2(\mathbb{F}_q), a + b\sqrt{d} \mapsto \begin{pmatrix} a & b \\ bd & a \end{pmatrix},$$

where $\sqrt{d}$ is a solution of $X^2 + d$. What goes wrong in characteristic 2? How to fix this?

4. Viewing $G = GL_2(\mathbb{F}_q)$ as the fixpoints of the Frobenius map F of $GL_2(\bar{\mathbb{F}}_q)$. Show that there exists $h \in GL_2(\mathbb{F}_{q^2})$ such that $h^{-1}h^q = w_0$.

5. Check for $G = GL_2(\mathbb{F}_q)$ the following equation (given by the general fact for finite groups that $\sum_{\chi \text{ irred. char.}} \chi(1)^2 = |G|$

$$(q-1) \cdot 1^2 + (q-1)q^2 + (\frac{q^2 - 3q + 2}{2})(q+1)^2 + (\frac{q^2 - q}{2})(q-1)^2 = |G|.$$

How did we obtain the characters used in this equation?

6. Show that there is exactly one W-orbit of Θ such that $\Theta \neq \Theta^{w_0}$ and $\Theta|_{\mathrm{SL}_2 \cap T} = \Theta^{w_0}|_{\mathrm{SL}_2 \cap T}$ (This also happens for the non-split torus).