Parameterisation of $\mathrm{Irr}(G^F)$

Setup: G conn. red alg group $/\overline{\mathbb{F}}_q$, F Steinberg map, G^F fin. grp. of Lie type over $\mathbb{F}_q$

So far: Machinery due to Deligne & Lusztig ('76):

T F-stable max. torus, $\theta \in \mathrm{Irr}(T) \rightsquigarrow R_T^G(\theta)$ virtual character.

Fact: For every $\rho \in \mathrm{Irr}(G^F)$ $\exists$ (T,θ) : $\langle R_T^G(\theta), \rho \rangle \neq 0$.

Remains: What are the multiplicities? How to parameterise $\mathrm{Irr}(G^F)$?

Langland's philosophy: "There exists a correspondence ~~between~~

Reps of red. groups $\longleftrightarrow$ Conj. class of Langlands-dual group"

$\rho_u \dashleftarrow\dashrightarrow u$

$\rho_s \dashleftarrow\dashrightarrow s$

Today: Jordan decomp. of $\mathrm{Irr}(G^F)$ due to Lusztig ('84, '88)

Construction: Consider the graph $DL(G^F)$ with vertices $\mathrm{Irr}(G^F)$ and edges $\{\rho_1,\rho_2\}$ if $\exists$ (T,θ) s.t. $\langle R_T^G(\theta), \rho_i\rangle \neq 0$ $i=1,2$.

Question: What are the connected components of $DL(G^F)$?

$\rightsquigarrow$ rational series / Lusztig series:

Last time: $(T,\theta)_{/G^F} \dashrightarrow (w,\theta)_{/G^F} \dashrightarrow (T^*, s)_{/G^{*F^*}}$ (write R_w for $R_w^{1_T}$)

$R_T^G(\theta)$ $\quad R_w^\theta$ $\quad R_{T^*}^G(s)$

$$\mathcal{E}(G^F,(s))_{G^{*F^*}} = \{\rho \in \mathrm{Irr}(G^F) \mid \langle R_{T^*}^G(s), \rho\rangle \neq 0 \text{ } F^*\text{-stable max torus } T^* \ni s\}$$ (Lusztig series / rational series)

$\simeq$ conn. components of $DL(G^F)$ ~~This is a theorem~~.

Def. / Thm Set of unipotent characters

$$\mathrm{Uch}(G^F) = \{\rho \in \mathrm{Irr}(G^F) \mid \langle R_T^G(1_T), \rho\rangle \neq 0 \text{ for } F\text{-stable max torus } T\}$$

$$= \{\rho \in \mathrm{Irr}(G^F) \mid \rho \text{ is geom. conj. to } 1_G\}$$

$$= \mathcal{E}(G,(1)) = \text{conn. comp. to which } 1_{G^F} \text{ belongs}$$

In total: Thm (Lusztig, ??)

$$\mathrm{Irr}(G^F) = \bigsqcup_{s} \mathcal{E}(G,(s))_{G^{*F^*}},$$

over semisimple conj. classes of G^{*F^*}.

Moreover, if $\rho \in \mathcal{E}(G^F,(s))$ then $\rho(1) = \dfrac{|G^F|_{p'}}{|H^{F^*}|} \sum_{T^*} \pm \langle R^G_{T^*}(s), \rho \rangle$,

where $H = C_{G^*}(s)$ and $T^* \subseteq H^\circ$ F-stable max torus.

Assume $Z(G)$ connected:

$\Longrightarrow$ $C_{G^*}(s)$ connected reductive algebraic group

$\mathcal{E}(G,(s)) = \mathcal{E}_{\lambda,n}$
$\simeq$ geom. conj. classes indexed by $\lambda \in X(T_0)$, n integer prime to p

Before we give the Jordan decomp. of characters we give a specified version for $\mathrm{Uch}(G^F)$.
Let W be the Weyl group of G. The root datum and F induce an automorphisms $\sigma \in \mathrm{Aut}(W)$ encoding how "twisted" G^F is.

Thm (Lusztig, 84'). ~~Let $Z(G)$ connected.~~ There exist a finite set $\overline{X}(W,\sigma)$ s.t.

$$\mathrm{Uch}(G^F) \xleftrightarrow{1-1} \overline{X}(W,\sigma), \quad \rho \longmapsto \bar{x}_\rho .$$

Moreover, there exist a collection of integers $\{m(w,\bar{x}) \mid w \in W, \bar{x} \in \overline{X}(W,\sigma)\}$ s.t. $\langle R_w, \rho \rangle = m(w,\bar{x}_\rho)$ $\forall w \in W$, $\rho \in \mathrm{Uch}(G^F)$ and

$$\rho(1) = \frac{1}{|W|} \sum_{w \in W} (-1)^{\ell(w)} m(w,\bar{x}_\rho) \frac{|G^F|_{q'}}{|T_w|}.$$

Rmk.
- $\overline{X}(w,\sigma)$ has a combinatorial description (Lusztig, 84') independent of q
- ~~this is~~
- Implementations exist in CHEVIE
- $\sigma = 1 \rightsquigarrow \overline{X}(W) \hookrightarrow \mathrm{Irr}(W)$

  W type A

Example: $G = GL_2$ (similar for $G = PGL_2$ since $G \to G/Z(G)$ induces an ~~iso~~ bijection between unipotent char.)

- $W = S_2 = \{1, s\}$, $\sigma = id$
- $\overline{X}(W,\sigma) = \overline{X}(W) = Irr(W) = \{\mathbb{1}, sgn\}$
- $m(w, \bar{x}_\rho) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$
- $\mathbb{1} \longleftrightarrow \mathbb{1}_{G^F}$: $R_1 = 1_G + St_G$, $R_s = 1_G - St_G$

have $|G^F| = (q^2-1)(q^2-q) = (q^2-1)q(q-1)$

$|T_1| = (q-1)^2$

$|T_s| = q^2 - 1$

$$\mathbb{1}(1) = \frac{1}{|W|} \sum_{w \in W} (-1)^{\ell(w)} m(w, \bar{x}_\rho) \frac{|G^F|_{q'}}{|T_w|}$$

$$= \frac{1}{2}\left(\frac{(q^2-1)(q-1)}{(q-1)^2} - \frac{\cancel{(q^2-1)}(q-1)}{\cancel{(q^2-1)}} \right)$$

$$= \frac{1}{2}\left(q+1 - (q-1) \right) = 1$$

- $sgn \longleftrightarrow St_G$: $\left(\langle St_G, Ind_{B^F}^{G^F}(1_B) \rangle \neq 0,\ St_G(1) = |G^F|_p \right)$

$$St_G(1) = \cancel{\frac{1}{|W|}} \cancel{\sum_{w \in W}} \frac{1}{2}\left(q+1 + (q-1) \right) = q = |G^F|_p$$

What about other irreducible characters?

Jordan decomposition of irred. characters

Thm (Lusztig, 84') $Z(G)$ conn., $s \in G^*$ semisimple s.t. $F^*(s) = s$, $H = C_{G^*}(s)$

$$\mathcal{E}(G^F,(s)) = \mathcal{E}(G^F,(s)_{G^{*F^*}}) \xleftrightarrow{1:1} \mathrm{Uch}(H^{F^*}),\ \rho \longleftrightarrow \rho_u$$

s.t. for any F^*-stable max. torus $T^* \subseteq H$, we have

$$\langle R^G_{T^*}(s), \rho \rangle = \varepsilon_G \varepsilon_H \langle R^H_{T^*}(1_{T^*}), \rho_u \rangle,$$ where $\varepsilon_G, \varepsilon_H \in \{-1,1\}$.

Rmk. There is also the notion of semisimple characters:
$\rho_s \in \mathrm{Irr}(G^F)$ unique character in $\mathcal{E}(G^F,(s))_{G^{*F^*}}$ with $\langle \rho, \Delta_G \rangle \neq 0$,
where
$$\Delta_G(g) = \begin{cases} |Z^\circ(G)^F|\, q^l &, g \in \mathcal{O}_0^F \\ 0 &, \text{else} \end{cases}$$
($\mathcal{O}_0$ unique conj. class of unipotent elements with $\overline{\mathcal{O}_0} = G_{uni}$)

Have: $\rho_s(1) = |G^{*F^*} : H^{F^*}|_{p'}$ and under the theorem above ρ_s corresponds to $1_{H^{F^*}}$

Example. Let $G = GL_2 \overset{Ex.}{\Rightarrow} G^* \cong GL_2 \cong G$

s.s. conj. classes:

(1) $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$
(a) $a = b \Rightarrow C_G(s) = G \rightsquigarrow$ already now $\mathrm{Uch}(G) = \{1, St_G\}$
(b) $a \neq b \Rightarrow C_G(s) = T_0 \rightsquigarrow \mathrm{Uch}(T_0) = \{1_{T_0}\}$

(2) $a, b \in \mathbb{F}_q$ s.t. $a \neq 0$ or $b \neq 0$ and $\sqrt{d} \notin \mathbb{F}_q$
$\begin{pmatrix} a & b \\ bd & a \end{pmatrix} \rightsquigarrow$ Ex.

What about G with $Z(G)$ not connected?

~~How to compute values of irreducible characters?, ~~ ~~Cha~~

~~what~~

General version of Jordan decomposition

Thm (Lusztig, 88')

Let $s \in G^*$ semisimple, $F^*(s) = s$, $H = C_{G^*}(s)$, $A(s)^{F^*} = \left(C_{G^*}(s) / C^\circ_{G^*}(s)\right)^{F^*}$.

Define
$$\mathcal{U}_{F^*}(H) = \left\{ (\mathcal{O}, a) \;\middle|\; \begin{array}{l} \mathcal{O} \text{ orbit of } A(s)^{F^*} \text{ on } \mathrm{Uch}(H^{\circ F^*}), \\ a \in \mathrm{Stab}_{A(s)^{F^*}}(x) \quad x \in \mathcal{O} \end{array} \right\}.$$

Then
$$\mathcal{E}(G^F, (s))_{G^{*F^*}} \longleftrightarrow \mathcal{U}_{F^*}(H), \quad \rho \longmapsto (\mathcal{O}_\rho, a_\rho),$$
s.t. for any F^*-stable maximal torus $T^* \subseteq H^\circ$, we have
$$\langle R^G_{T^*}(s), \rho \rangle = \varepsilon_G \varepsilon_H \sum_{i=1}^{r} \langle R^{H^\circ}_{T^*}(\mathbb{1}), \rho_i \rangle \qquad \mathcal{O}_\rho = \{\rho_1, \dots, \rho_r\}.$$

Rmk. If $A(s)^{F^*} = \{1\}$ $\Rightarrow$ $\mathcal{U}_{F^*}(H) \longleftrightarrow \mathrm{Uch}(H^{\circ F^*})$
$$(\psi, 1) \longleftrightarrow \psi$$

$\left(\text{if } \gcd(|s|, |Z/Z^\circ)_F|) = 1 \Rightarrow A(s)^{F^*} = \{1\} \right)$

Example $G^F = SL_4(q)$, $\Rightarrow G^{*F^*} = PGL_4(q)$

q odd

Have $|Z(G^F)| = \begin{cases} 2 & , 4 \nmid q+1 \\ 4 & , 4 \mid q-1 \end{cases}$

Consider $s = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$, $a = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{pmatrix} \in GL_4(q)$ and $\bar{s}$ image in G^*

$H = C_{G^*}(s) \Rightarrow H^\circ$ has type $A_1 \times A_1$, $\dim Z(H^\circ) = 1$, $H/H^\circ \cong C_2 = \langle a \rangle$

Have $\mathrm{Uch}(H^\circ) = \{ \psi_1, \psi_2, \psi_2', \psi_3 \}$

$\psi_1 = \mathbb{1}$, ψ_2, ψ_2' degree q, $\psi_3 = St_{H^\circ}$ degree q^2

For H°-conj. classes of F^*-stable max. tori in H°: $T_1^*, T_2^*, T_2'^*, T_3^*$ with $|T_1^*| = (q-1)^3$, $|T_2^*| = |T_2'^*| = (q-1)^2(q+1)$, $|T_3^*| = (q-1)(q+1)^2$

Can compute: $R^{H^\circ}_{T_1^*}(1_{T_1^*}) = \psi_1 + \psi_2 + \psi_2' + \psi_3$

$R^{H^\circ}_{T_2^*}(1_{T_2^*}) = \psi_1 + \psi_2 - \psi_2' - \psi_3$

$R^{H^\circ}_{T_2'^*}(1_{T_2'^*}) = \psi_1 - \psi_2 + \psi_2' - \psi_3$

$R^{*}_{T_3^*}(1_{T_3^*}) = \psi_1 - \psi_2 - \psi_2' + \psi_3$

Action of $\bar{a}$ on H° exchanges components of type A_1 $\rightsquigarrow$ swaps $T_2^*, T_2'^*$, fixes T_1^*, T_3^*

Similarly, swaps ψ_2, ψ_2' and fixes ψ_1, ψ_3

$\mathcal{O}_1 = [\psi_1]$, $\mathcal{O}_2 = \{\psi_2, \psi_2'\}$, $\mathcal{O}_3 = \{\psi_3\}$

$\Rightarrow \mathcal{U}_{F^*}(H) = \{(\mathcal{O}_1, 1), (\mathcal{O}_1, \bar{a}), (\mathcal{O}_2, 1), (\mathcal{O}_3, 1), (\mathcal{O}_3, \bar{a})\}$

$\Rightarrow |\mathcal{E}(G, \bar{s})| = 5$, let $\rho \in \mathcal{E}(G, \bar{s})$ correspond to $(\mathcal{O}_2, 1)$

$\langle R^H_{T_1^*}(\bar{s}), \rho\rangle = \pm(\langle R^{H^\circ}_{T_1^*}(1_{T_1^*}), \psi_2\rangle + \langle R^{H^\circ}_{T_1^*}(1_{T_1^*}), \psi_2'\rangle) = \pm 2$

$\quad = \pm$ ... $= 0$

In this case + ... $= 0$

2 ... $= -2$

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What about character values?

Prop (DL, 76') $\rho \in \operatorname{Irr}(G^F)$, s semisimple $\rho(s) = \dfrac{1}{|C^\circ_G(s)^F|_p} \sum_{(T,\theta)} \varepsilon_G \varepsilon_T \langle R^G_T(\theta), \rho\rangle\, \theta(s)$

sum over all pairs (T, θ) s.t. $T \leq G$ F-stable max. torus, $s \in T$, $\theta \in \operatorname{Irr}(T^F)$

What about values on unipotent conj classes?

Lusztig 85-86 $\rightsquigarrow$ Character Sheaves

~~Appendix: relative F-rank~~

Relative F-ranks

F induces an endomorphism $\varphi : X(T) \to X(T)$

$\rightsquigarrow$ extend to $\mathbb{R}$ i.e. $\varphi : X(T)_{\mathbb{R}} \to X(T)_{\mathbb{R}}$, $X(T)_{\mathbb{R}} = \mathbb{R} \otimes_{\mathbb{Z}} X(T)$

r = relative F-rank of $T := \dim [v \mid \varphi_{\mathbb{R}}(v) = qv]$ $\rightsquigarrow \varepsilon_T = (-1)^r$

If $T_0 \leq G$ max. split torus, set $\varepsilon_G = \varepsilon_{T_0}$.

Have: relative F-rank of $G = \max\{ i \geq 0 \mid (q-1)^i \mid |G(\mathbb{F}_q)| \}$

Dual groups (G,F), (G^*,F^*) with G, G^* conn. red. and F, F^* Steinberg maps.

We say (G,F) and (G^*,F^*) are in duality if G^* has dual root datum with adjoint Frobenius action. More specifically:

There exist max. split tori $T_0 \subseteq G$, $T_0^* \subseteq G^*$ s.t. the root datum of G $(X, \Phi, Y, \Phi^\vee)$ and the root datum of G^* $(X^*, \Phi^*, Y^*, \Phi^{*\vee})$ satisfy (w.r.t. T_0, T_0^*)

(a) $\exists\ \delta : X \xrightarrow{\sim} Y^*$ s.t. $\delta(R) = \Phi^{*\vee}$ and

$$\langle \lambda, \alpha^\vee \rangle = \langle \alpha^*, \delta(\lambda) \rangle \quad \forall \lambda \in X(T_0), \alpha \in \Phi,$$

where $\alpha^* \in \Phi^*$ with $\delta(\alpha) = \alpha^{*\vee}$.

(b) $\delta(\lambda \circ F|_{T_0}) = F^*|_{T_0^*} \circ \delta(\lambda) \quad \forall \lambda \in X.$