Deligne-Lusztig theory notes week 3

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Layout: Sections 1 and 2 continue the discussion of Frobenius maps on algebraic groups from last week. Sections 3 and 4 establish some background on finite groups. Sections 5, 6 and 7 cover Harish-Chandra theory. Section 8 has some additional comments on the references used.

Throughout let p be a prime, q a power of p, $\mathbf{G}$ a connected reductive algebraic group over $\mathbb{F}_q$, and $\Bbbk$ an algebraically closed field of characteristic zero. For a set X and function $F : X \to X$ we write X^F for the subset of elements fixed by F, that is $\{x \in X \mid F(x) = x\}$.

1 Steinberg maps and finite groups of Lie type

Recall that a q-Frobenius map on an algebraic group $\mathbf{G}$ is a group variety morphism $F : \mathbf{G} \to \mathbf{G}$ corresponding to an algebra morphism $F^* : \mathcal{O}_{\mathbf{G}} \to \mathcal{O}_{\mathbf{G}}$ such that

1. $F^*(\mathcal{O}_{\mathbf{G}}) = \{f^q \mid f \in \mathcal{O}_{\mathbf{G}}\}$, and
2. for every $f \in \mathcal{O}_{\mathbf{G}}$ there exists some $m \in \mathbb{N}$ with $(F^*)^m(f) = f^{q^m}$.

We then say that F gives $\mathbf{G}$ an $\mathbb{F}_q$-rational structure. We call F a *Steinberg map* or *Frobenius root* if some positive power of F is a Frobenius map. The group $\mathbf{G}^F$ is a finite group, and such groups are called finite groups of Lie type.

We say a subvariety $\mathbf{V} \subseteq \mathbf{G}$ is *F-stable* if $F(\mathbf{V}) \subseteq \mathbf{V}$. If we have $\mathbf{T} \subseteq \mathbf{B} \subseteq \mathbf{G}$ with $\mathbf{T}$ an F-stable maximal torus and $\mathbf{B}$ and F-stable Borel subgroup, then [DM] calls $\mathbf{T}$ *quasi-split*.

Proposition (1.4.12 in [GM] or 4.2.22 in [DM]). *Let $\mathbf{G}$ be connected reductive and $F : \mathbf{G} \to \mathbf{G}$ a Steinberg map. All F-stable torus and Borel pairs are $\mathbf{G}^F$-conjugate. That is, if $\mathbf{T} \subseteq \mathbf{B}$ and $\mathbf{T}' \subseteq \mathbf{B}'$ are two pairs of an F-stable torus and F-stable Borel subgroup of $\mathbf{G}$, then there is $g \in \mathbf{G}$ with $F(g) = g$, $g\mathbf{T}g^{-1} = T'$ and $g\mathbf{B}g^{-1} = \mathbf{B}'$.*

Recall that $\mathbf{G}$ has a Weyl group $\mathbf{W}$ which is a Coxeter group with generating set S. For $I \subseteq S$, we write $\mathbf{W}_I$ for the subgroup generated by I. For $w \in \mathbf{W}$ we write $\ell(w)$ for the length of w, meaning the minimal length of w when written as a product of generators. When $\mathbf{G}$ has quasi-split torus, pretty much every aspect of the theory of reductive groups descends to the finite group $\mathbf{G}^F$:

Proposition (4.4.1, 4.4.8 in [DM]). *Let $\mathbf{G}$ be connected reductive, $F : \mathbf{G} \to \mathbf{G}$ a Steinberg map and $\mathbf{T} \subseteq \mathbf{B} \subseteq \mathbf{G}$ an F-stable torus and Borel pair. Then:*

1. *The Weyl group $\mathbf{W} = N_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$ satisfies $\mathbf{W}^F = N_{\mathbf{G}}(\mathbf{T})^F/\mathbf{T}^F$.*
2. *$(\mathbf{B}^F, N_{\mathbf{G}}(\mathbf{T})^F)$ is a BN-pair for $\mathbf{G}^F$ with Weyl group $\mathbf{W}^F$.*
3. *The action of F on $\mathbf{W}$ is an automorphism of the Coxeter system $(\mathbf{W}, S)$. In particular, F permutes the elements of S, and we write S/F for the set of F-orbits.*

4. $\mathbf{W}^F$ *is a Coxeter group with generating set* $\{w_I \mid I \in S/F\}$*, where* w_I *is the longest element of* $\mathbf{W}_I$*.*

5. *We have a Bruhat decomposition* $\mathbf{G}^F = \bigsqcup_{w \in \mathbf{W}^F} \mathbf{B}^F w \mathbf{B}^F$*.*

Beware that $\mathbf{W}^F$ can be different from $\mathbf{W}$. Also, a priori $N_{\mathbf{G}}(\mathbf{T})^F = N_{\mathbf{G}^F}(\mathbf{T})$ could be smaller than $N_{\mathbf{G}^F}(\mathbf{T}^F)$, although I don't know of an example of this.

2 Examples of Frobenius and Steinberg maps

Recall that $\mathbb{F}_q$, the finite field of order q, can be defined as the field extension of $\mathbb{F}_p$ consisting of the q distinct roots of the polynomial $x^q - x$. This means that if F_q is the field homomorphism $F_q(a) = a^q$ on the algebraic closure $\overline{\mathbb{F}}_p$, then we have $(\overline{\mathbb{F}}_p)^{F_q} = \mathbb{F}_q$. Now, $\mathbb{F}_{q^2}$ is a 2-dimensional vector space over $\mathbb{F}_q$, so every element is an $\mathbb{F}_q$-linear combination of 1 and λ where λ is some fixed element of $\mathbb{F}_{q^2}$ with $\lambda \notin \mathbb{F}_q$. We can then think of the action of F_q on $\mathbb{F}_{q^2}$ as 'conjugation'; it sends $a + b\lambda$ to $a^q + b^q\lambda^q = a + b\lambda^q$. This is analogous to conjugation in $\mathbb{C}$, and in some circumstances it's almost identical. For example, if $q = 3$ then $2 = -1$ does not have a square root in $\mathbb{F}_3$, so we take λ to be a square root of -1 in $\mathbb{F}_9$. Then $F_3(a + b\lambda) = a + b\lambda^3 = a - b\lambda$.

If $\mathbf{G} = \mathrm{GL}_n(\overline{\mathbb{F}}_p)$ and F is the morphism raising matrix entries to the power of q, then F is a Frobenius map and $\mathbf{G}^F = \mathrm{GL}_n(\mathbb{F}_q)$. The Borel subgroup $\mathbf{B}$ of upper triangular matrices and the maximal torus $\mathbf{T}$ of diagonal matrices are then F-stable subgroups, the groups $\mathbf{B}^F$ and $\mathbf{T}^F$ are the upper triangular and diagonal matrices with entries in $\mathbb{F}_q$, and $\mathbf{W}^F = \mathbf{W}$. A less trivial example of a Frobenius map on GL_n is $F'(M) = F((M^\top)^{-1})$, where $M^\top$ is the transpose of the matrix M, which satisfies $(F')^2 = F^2$ (see [DM, 4.3.3]). The group $\mathrm{GL}_n^{F'}$ is also denoted $\mathrm{GU}_n(q)$, the *general unitary group*, and it is the group of unitary transformations M of $(\mathbb{F}_{q^2})^n$. A matrix M is unitary if $MM^* = I_n$ where M^* is the conjugate transpose, using the adjusted notion of 'conjugation' above. This is analogous to the group of unitary matrices over $\mathbb{C}$.

The subgroup of upper triangular matrices is not F'-stable, but fortunately there is an alternate presentation of GU_n where it is. Let $F'' = F(J_n(M^\top)^{-1}J_n)$ where J_n is the matrix with 1's along the top-right to bottom-left diagonal (see [GM, 1.3.19]), and this has the nice property that the subgroups $\mathbf{B}$ and $\mathbf{T}$ are F''-stable. Recall that the Weyl group $\mathbf{W}$ of GL_n is isomorphic to the symmetric group, with generators $s_1, \dots s_{n-1}$ where

$$s_i = \begin{pmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & \ddots & & & & \\ & & & \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} & & & \\ & & & & \ddots & & \\ & & & & & 1 & \\ & & & & & & 1 \end{pmatrix} \quad \text{with } \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \text{ in rows } i, i+1.$$

The Steinberg map F'' acts on W by sending s_i to s_{n-i} (if you know what a Coxeter diagram or Dynkin diagram is, F'' is the diagram automorphism in type A_{n-1}). Consequently, $\mathbf{W}^{F''}$ will be a Coxeter group with rank $\lceil n/2 \rceil$ and generators $s_1 s_{n-1}, s_2 s_{n-2}, \dots, s_m$ if $n = 2m$ or $s_m s_{m+1} s_m$ if $n = 2m+1$.

If F is a Frobenius map on $\mathbf{G}$ and r is a positive integer, then we can construct the map

$$\mathbf{G}^r \to \mathbf{G}^r, \quad (g_1, g_2, \dots, g_r) \mapsto (F(g_r), g_1, \dots, g_{r-1}).$$

This is an example of a Steinberg map that is not a Frobenius map (see [GM, Example 1.4.23, 1.4.29]).

3 Character theory

This section is a brief recap of character theory for finite groups.

If G is a finite group and V is a representation of G, then we can write the action of G as a group homomorphism $\rho : G \to \mathrm{GL}(V)$. The *character* of V is the function $\chi : G \to \Bbbk$ sending $g \in G$ to $\mathrm{tr}(\rho(g))$, the trace of the action matrix of g.

Example. The character of the trivial representation of G sends every element to 1. The character of the regular representation $\Bbbk G$ sends the group identity to $|G|$ and every other element to 0.

Here are some important properties of characters:

- $\chi_V(1) = \dim V$
- $\chi_{V \oplus W} = \chi_V + \chi_W$
- $\chi_{V \otimes W} = \chi_V \cdot \chi_W$
- $\chi_{V^*} = \overline{\chi_V}$ where the bar means the complex conjugate.
- Two representations are isomorphic if and only if they have the same character (assuming $\Bbbk$ is algebraically closed and has characteristic 0).
- χ_V is a class function, meaning $\chi_V(g) = \chi_V(hgh^{-1})$ for all $g, h \in G$.
- The space of class functions $G \to \Bbbk$ is a vector space, and the characters of the irreducible representations form a basis for the space of class functions. A linear combination of irreducible characters with integer coefficients is called a *virtual character*.
- We define an inner product on the space of class functions given by
$$\langle \chi, \xi \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g) \overline{\xi(g)}.$$
If χ, ξ are irreducible characters, we have $\langle \chi, \xi \rangle = 1$ if $\chi = \xi$ and 0 otherwise, and so
$$\langle \chi_V, \chi_W \rangle = \dim \mathrm{Hom}_G(V, W).$$

4 Tensor-Hom adjunction and induction

This section is an overview of induction, restriction, inflation and invariants for finite groups.

Let R be a (unital associative) ring, M a right R-module and N a left R-module. The *tensor product* $M \otimes_R N$ is the quotient of $\mathbb{Z}[M \times N]$ by the relations $mr \otimes n = m \otimes rn$ for $m \in M, r \in R, n \in N$. Alternately, $M \otimes_R N$ is the universal abelian group such that if X is an abelian group and $f : M \times N \to X$ is a group morphism with $f(mr, n) = f(m, rn)$ for all $m \in M, r \in R, n \in N$, then there is a unique group morphism $f' : M \otimes_R N \to X$ such that $f(m, n) = f'(m \otimes n)$.

Now, $\mathrm{Hom}(M, X)$ has a left action of R given by $r \cdot f = f(- \cdot r)$. Given a function $f : M \times N \to X$ as above, the function $n \mapsto f(-, n)$ is an element of $\mathrm{Hom}_R(N, \mathrm{Hom}(M, X))$. The universal property of the tensor product then means that we have an isomorphism

$$\begin{aligned} \mathrm{Hom}(M \otimes_R N, X) &\cong \mathrm{Hom}_R(N, \mathrm{Hom}(M, X)) \\ f &\mapsto (n \mapsto f(-, n)), \\ (m \otimes n \mapsto f(n)(m)) &\leftarrow\!\shortmid f. \end{aligned}$$

If M and X also have a left action of some other module S, then similarly we have an isomorphism

$$\mathrm{Hom}_S(M \otimes_R N, X) \cong \mathrm{Hom}_R(N, \mathrm{Hom}_S(M, X)).$$

This is called *tensor-hom adjunction*, because it is an adjunction between the functors

$$M \otimes_R - : R\text{-mod} \to S\text{-mod on the left and}$$
$$\mathrm{Hom}(M, -) : S\text{-mod} \to R\text{-mod on the right.}$$

If $R = \Bbbk G$ and $S = \Bbbk H$ are the group algebras of some finite groups G and H, then an adjunction $L \dashv R$ between two functors $L : \mathrm{Rep}G \to \mathrm{Rep}H$ and $R : \mathrm{Rep}H \to \mathrm{Rep}G$ can be understood on the level of characters as

$$\langle L\chi_1, \chi_2 \rangle_H = \langle \chi_1, R\chi_2 \rangle_G$$

where χ_1 is a character of G and χ_2 is a character of H. Next we'll look at some examples of these functors between representations of groups and use tensor-hom to find some adjunctions between them.

Scenario 1: subgroup. Suppose H is a subgroup of a finite group G. For a representation V of H and an element $g \in G$, the conjugate gHg^{-1} is also a subgroup of G, and we write gV for the conjugate representation with action $ghg^{-1} \cdot gv = g(h \cdot v)$. In the literature, these are sometimes denoted ${}^gH$ and ${}^gV$. We have 2 exact functors between the categories $\mathrm{Rep}G$ and $\mathrm{Rep}H$:

1. The *restriction functor* $\mathrm{Res}^G_H : \mathrm{Rep}G \to \mathrm{Rep}H$ sends a G-representation V to its underlying H-representation. We can write this in a number of ways:

$$\begin{array}{ccccl} \mathrm{Res}^G_H(V) & \cong & \Bbbk G \otimes_{\Bbbk G} V & \cong & \mathrm{Hom}_G(\Bbbk G, V) \text{ with action } h \cdot f = f(- \cdot h) \\ v & \leftrightarrow & 1 \otimes v & \leftrightarrow & (g \mapsto g \cdot v) \end{array}$$

2. The *induction functor* $\mathrm{Ind}^G_H : \mathrm{Rep}H \to \mathrm{Rep}G$ sends an H-representation V to $\bigoplus_i g_i V$ where $1 = g_1, g_2, \ldots, g_{|G|/|H|}$ are H-coset representatives, and the action is $g \cdot g_i v = g_j(h \cdot v)$ where $h \in H$ and g_j are the unique elements with $gg_i = g_j h$. We have

$$\begin{array}{ccccl} \mathrm{Ind}^G_H(V) & \cong & \Bbbk G \otimes_{\Bbbk H} V & \cong & \mathrm{Hom}_H(\Bbbk G, V) \text{ with action } g \cdot f = f(- \cdot g) \\ \sum_i g_i v & \leftrightarrow & \sum_i g_i \otimes v & \mapsto & (hg_i^{-1} \mapsto h \cdot v_i) \\ \sum_i g_i f(g_i^{-1}) & \leftrightarrow & \sum_i g_i \otimes f(g_i^{-1}) & \leftarrow & f \end{array}$$

(Note that $\mathrm{Hom}_H(\Bbbk G, V)$ is sometimes called *coinduction*, which for general algebra modules is not necessarily isomorphic to induction.)

Scenario 2: quotient. Suppose N is a normal subgroup of G and $H = G/N$, and write ϕ for the projection map $G \to H$. The H-module $\Bbbk H$ can then be interpreted as a left or right G-module with action $g \cdot h = \phi(g)h$ or $h \cdot g = h\phi(g)$. We define $e_N \in \Bbbk G$ by

$$e_N = \frac{1}{|N|} \sum_{n \in N} n, \quad \text{so we have:} \quad \begin{array}{rcl} e_N^2 & = & e_N, \\ ne_N & = & e_N \text{ for } n \in N, \\ ge_N & = & e_N g \text{ for } g \in G. \end{array}$$

We again have 2 exact functors between the categories $\mathrm{Rep}G$ and $\mathrm{Rep}H$:

1. The *inflation* (or *lifting*) *functor* $\mathrm{Inf}_H^G : \mathrm{Rep}H \to \mathrm{Rep}G$ sends an H-representation V to the same vector space with G action $g \cdot v = \phi(g) \cdot v$. We can write this in a number of ways:

$$\begin{array}{ccccl} \mathrm{Inf}_H^G(V) & \cong & \Bbbk H \otimes_{\Bbbk H} V & \cong & \mathrm{Hom}_H(\Bbbk H, V) \text{ with action } g \cdot f = f(- \cdot \phi(g)) \\ v & \leftrightarrow & 1 \otimes v & \leftrightarrow & (h \mapsto h \cdot v) \end{array}$$

   If H is also a subgroup of G so that we have a semi-direct product decomposition $G = H \ltimes N$, then we also have $\mathrm{Inf}_H^G(V) \cong e_N \otimes_{\Bbbk H} V = (e_N \Bbbk H) \otimes_{\Bbbk H} V$ with $v \mapsto e_N \otimes v$. This is because any $g \in G$ can be written as $g = hn$ with $h = \phi(g) \in H$ and $n \in N$, so

$$g \cdot e_N \otimes v = hne_N \otimes v = he_N \otimes v = e_N h \otimes V = e_N \otimes (h \cdot v).$$

2. The *invariants functor* $\mathrm{Inv}_H^G : \mathrm{Rep}G \to \mathrm{Rep}H$ sends an G-representation V to the subspace $V^N = \{v \in V \mid n \cdot v = v \; \forall n \in N\}$. The action of $\phi(g) \in H$ is $\phi(g) \cdot v = g \cdot v$, which is well-defined since if $\phi(g) = \phi(g')$ then $g = g'n$ for some $n \in N$, and $n \cdot v = v$. We have

$$\begin{array}{ccccccl} \mathrm{Inv}_H^G(V) & \cong & e_N V & \cong & \Bbbk H \otimes_{\Bbbk G} V & \cong & \mathrm{Hom}_G(\Bbbk H, V) \text{ with } h \cdot f = f(- \cdot h) \\ v & \leftrightarrow & e_N v & \leftrightarrow & 1 \otimes v & \mapsto & (- \cdot v) \\ e_N f(1) & \leftrightarrow & e_N f(1) & \leftarrow\!\shortmid & 1 \otimes f(1) & \leftarrow\!\shortmid & f \end{array}$$

   (Note that $\mathrm{Hom}_H(\Bbbk G, V) = V_N$ is called *coinvariants*, which for general algebra modules is not necessarily isomorphic to invariants.)

Using these different expressions and tensor-hom adjunction, we obtain the following bi-adjunctions.

Proposition. *For finite groups G and H:*

- *When $H \subseteq G$, Ind_H^G is both left and right adjoint to Res_H^G.*
- *When $H = G/N$, Inv_H^G is both left and right adjoint to Inf_H^G.*

Suppose H and K are subgroups of G. For $g \in G$, the set $KgH = \{kgh \mid k \in K, h \in H\}$ is called a *double coset*. The set of double cosets is denoted $K \backslash G / H$, and all the double cosets are disjoint (since if $k_1 g_1 h_1 = k_2 g_2 h_2$ then $g_1 = k_1^{-1} k_2 g_2 h_2 h_1^{-1} \in K g_1 H$). Note that, unlike single cosets, double cosets can be very asymmetrical. For example, if $G = S_3$, $H = \{1, (12)\}$ and $K = \{1, (23)\}$ then the double cosets are $\{1, (12), (23), (132)\}$ and $\{(13), (123)\}$.

Now, let $s_1, \ldots, s_n$ be a set of double coset representatives (so $G = \bigsqcup K s_i H$), and for each i choose representatives $k_{i,j}$ for the H-cosets comprising $K s_i H$. That is, $\{k_{i,j} \mid j\}$ is a family of coset representatives for $(s_i H s_i^{-1} \cap K) \subseteq K$. Then we have

$$\mathrm{Res}_K^G \mathrm{Ind}_H^G V = \bigoplus_{i=1}^n \bigoplus_j k_{i,j} s_i V = \bigoplus_{i=1}^n \mathrm{Ind}_{s_i H s_i^{-1} \cap K}^K \mathrm{Res}_{s_i H s_i^{-1} \cap K}^{s_i H s_i^{-1}} (s_i V).$$

This is called the *Mackey formula*.

5 Harish-Chandra induction and restriction

Let $\mathbf{G}$ be a connected reductive group over $\overline{\mathbb{F}}_p$ and F a Steinberg map on $\mathbf{G}$. Let $\mathbf{P}$ be an F-stable parabolic subgroup of $\mathbf{G}$ and $\mathbf{L}$ an F-stable Levi subgroup of $\mathbf{P}$ so that we have an F-stable Levi decomposition $\mathbf{P} = \mathbf{L} \ltimes \mathbf{U}$.

We define exact bi-adjoint functors called Harish-Chandra induction and restriction:

$$R^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}} = \mathrm{Ind}^{\mathbf{G}^F}_{\mathbf{P}^F} \circ \mathrm{Inf}^{\mathbf{P}^F}_{\mathbf{L}^F} \quad \text{and} \quad {}^*R^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}} = \mathrm{Inv}^{\mathbf{P}^F}_{\mathbf{L}^F} \circ \mathrm{Res}^{\mathbf{G}^F}_{\mathbf{P}^F}.$$

Here are some more ways of writing these:

$$\begin{array}{rcccccc} R^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}}(V) & \cong & \Bbbk\mathbf{G}^F \otimes_{\Bbbk\mathbf{P}^F} \mathrm{Inf}^{\mathbf{P}^F}_{\mathbf{L}^F}(V) & \cong & \Bbbk(\mathbf{G}^F/\mathbf{U}^F) \otimes_{\Bbbk\mathbf{L}^F} V & \cong & (\Bbbk\mathbf{G}^F e_{\mathbf{U}^F}) \otimes_{\Bbbk\mathbf{L}^F} V \\ & & g \otimes v & \leftrightarrow & (g\mathbf{U}^F) \otimes v & \leftrightarrow & (ge_{\mathbf{U}^F}) \otimes v \end{array}$$

$$\begin{array}{rcccc} {}^*R^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}}(V) & \cong & V^{\mathbf{U}_F} = \{v \in V \mid u \cdot v = v \ \forall u \in \mathbf{U}^F\} & \cong & e_{\mathbf{U}^F} V \\ & & v & \leftrightarrow & (g\mathbf{U}^F) \otimes v \end{array}$$

where $e_{\mathbf{U}^F} = \frac{1}{|\mathbf{U}^F|} \sum_{u \in \mathbf{U}^F} u$ and the right action of l on $\Bbbk(\mathbf{G}^F/\mathbf{U}^F)$ is $g\mathbf{U}^F \cdot l = gl\mathbf{U}^F$, which is well-defined since $\mathbf{L}$ normalises $\mathbf{U}$. These R functors are transitive, in the sense that if $\mathbf{M} \subset \mathbf{Q}$ are F-stable Levi and parabolic subgroups contained in $\mathbf{L}$ and $\mathbf{P}$ respectively then

$$R^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}} \circ R^{\mathbf{L}}_{\mathbf{M}\subseteq\mathbf{L}\cap\mathbf{Q}} = R^{\mathbf{G}}_{\mathbf{M}\subseteq\mathbf{P}} \text{ and } {}^*R^{\mathbf{L}}_{\mathbf{M}\subseteq\mathbf{L}\cap\mathbf{Q}} \circ {}^*R^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}} = {}^*R^{\mathbf{G}}_{\mathbf{M}\subseteq\mathbf{P}}$$

We also obtain a version of the Mackey formula when $\mathbf{M} \subset \mathbf{Q}$ are arbitrary F-stable Levi and parabolic subgroups of $\mathbf{G}$:

Theorem (5.2.1 in [DM])**.** *We have*

$${}^*R^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}} \circ R^{\mathbf{G}}_{\mathbf{M}\subseteq\mathbf{Q}}(V) = \bigoplus_x R^{\mathbf{L}}_{(\mathbf{L}\cap x\mathbf{M}x^{-1})\subseteq(\mathbf{L}\cap x\mathbf{Q}x^{-1})} \circ {}^*R^{x\mathbf{M}x^{-1}}_{(\mathbf{L}\cap x\mathbf{M}x^{-1})\subseteq(\mathbf{P}\cap x\mathbf{M}x^{-1})}(xV)$$

where x ranges over representatives of the double cosets $\mathbf{L}^F \backslash \mathcal{S}(\mathbf{L}, \mathbf{M})^F / \mathbf{M}^F$, where

$$\mathcal{S}(\mathbf{L}, \mathbf{M}) = \{x \in \mathbf{G} \mid \mathbf{L} \cap x\mathbf{M}x^{-1} \text{ contains a maximal torus of } \mathbf{G}\}.$$

Note that the natural map $\mathbf{L}\backslash\mathcal{S}(\mathbf{L}, \mathbf{M})/\mathbf{M} \to \mathbf{P}\backslash\mathbf{G}/\mathbf{Q}$ given by $\mathbf{L}x\mathbf{M} \mapsto \mathbf{P}x\mathbf{Q}$ is an isomorphism [DM, Lemma 5.2.2], so really this is a sum over $(\mathbf{P}\backslash\mathbf{G}/\mathbf{Q})^F = \mathbf{P}^F\backslash\mathbf{G}^F/\mathbf{Q}^F$ with the extra assumption that the representatives x be chosen such that the intersections $\mathbf{L} \cap x\mathbf{M}x^{-1}$ contain maximal tori, so that the R functors are well-defined.

Example. Take $\mathbf{L} = \mathbf{T}$ to be a maximal torus. Then $\mathbf{T} \cap x\mathbf{T}x^{-1}$ contains a maximal torus if and only if $x \in N_{\mathbf{G}}(\mathbf{T})$, and hence $\mathbf{T}\backslash\mathcal{S}(\mathbf{T}, \mathbf{T})/\mathbf{T} = N_{\mathbf{G}}(\mathbf{T})/\mathbf{T} = \mathbf{W}$. So, if $\mathbf{B}, \mathbf{B}'$ are F-stable Borel subgroups containing $\mathbf{T}$ and χ, χ' are characters of $\mathbf{T}$, then by adjunction and the Mackey formula we have

$$\langle R^{\mathbf{G}}_{\mathbf{T}\subseteq\mathbf{B}}\chi, R^{\mathbf{G}}_{\mathbf{T}\subseteq\mathbf{B}'}\chi' \rangle_{\mathbf{G}^F} = \langle \chi, {}^*R^{\mathbf{G}}_{\mathbf{T}\subseteq\mathbf{B}} \circ R^{\mathbf{G}}_{\mathbf{T}\subseteq\mathbf{B}'}\chi' \rangle_{\mathbf{T}^F} = \sum_{w \in \mathbf{W}^F} \langle \chi, x_w\chi' \rangle_{\mathbf{T}^F}$$

where $x_w \in N_{\mathbf{G}}(\mathbf{T})^F$ is a representative of w and $x\chi$ is the character $x\chi(g) = \chi(xgx^{-1})$. This gives an explicit formula for the dimensions of hom spaces between representations $\mathbf{R}^{\mathbf{G}}_{\mathbf{T}}V$. One can extend this result to arbitrary $\mathbf{L}$ and use it prove the following theorem:

Theorem (5.3.1 in [DM])**.** *The character of $R^{\mathbf{G}}_{\mathbf{L}\subseteq\mathbf{P}}(V)$ is independent of the parabolic subgroup $\mathbf{P}$ containing $\mathbf{L}$. Consequently, we shorten the notation to $R^{\mathbf{G}}_{\mathbf{L}}$.*

If $\mathbf{L}$ is an F-stable Levi subgroup of an F-stable parabolic subgroup of $\mathbf{G}$, and Λ is a simple $\mathbf{L}^F$-representation, then we call $(\mathbf{L}, \Lambda)$ a *cuspidal pair* if one of the following conditions are satisfied (these conditions are equivalent by adjunction):

1. If $\mathbf{L}' \subseteq \mathbf{L}$ is an F-stable Levi subgroup of an F-stable parabolic subgroup of $\mathbf{L}$, V is a simple $(\mathbf{L}')^F$-representation, and there exists a surjective morphism $R^{\mathbf{L}}_{\mathbf{L}'}V' \twoheadrightarrow \Lambda$, then $\mathbf{L} = \mathbf{L}'$ and $V \cong \Lambda$.

2. If $\mathbf{L}' \subseteq \mathbf{L}$ is an F-stable Levi subgroup of an F-stable parabolic subgroup of $\mathbf{L}$, then ${}^*R^{\mathbf{L}}_{\mathbf{L}'}\Lambda = 0$.

The *Harish-Chandra series* $\operatorname{Simp}(\mathbf{G}^F \mid (\mathbf{L}, \Lambda))$ of a cuspidal pair $(\mathbf{L}, \Lambda)$ is the set of simple quotients of $R^{\mathbf{G}}_{\mathbf{L}}\Lambda$, or equivalently the set of all $\mathbf{G}^F$-representations V such that Λ is a submodule of ${}^*R^{\mathbf{G}}_{\mathbf{L}}(V)$.

Theorem (5.3.7 in [DM])**.** *If $(\mathbf{L}, \Lambda)$ and $(\mathbf{L}', \Lambda')$ are cuspidal pairs, then they have the same series if they are $\mathbf{G}^F$-conjugate (meaning there exists $x \in \mathbf{G}^F$ with $\mathbf{L}' = x\mathbf{L}x^{-1}$ and $\Lambda' = x\Lambda$), and otherwise have disjoint series. Consequently, the set of simples $\operatorname{Simp}(\mathbf{G}^F)$ partitions into Harish-Chandra series.*

For fixed $\mathbf{L}$, the *Harish-Chandra series associated to* $\mathbf{L}$ is the union of the Harish-Chandra series of all cuspidal $(\mathbf{L}, \Lambda)$. If $\mathbf{L}$ is a quasi-split torus then all representations of $\mathbf{L}^F$ are cuspidal, and this series is called the *principal series*. If $\mathbf{L} = \mathbf{G}$ then the series consists only of cuspidal representations and is called the *discrete series*. So in theory, we can split the classification of representations of $\mathbf{G}^F$ into two steps: compute the discrete series of any $\mathbf{L} \subseteq \mathbf{G}$ contained in an F-stable parabolic, then compute the Harish-Chandra series on these representations. In practice, the discrete series representations are the hardest ones to obtain. However, the second step is something we can build up a theory for.

6 Classifying simples in $R^{\mathbf{G}}_{\mathbf{T}}\mathbb{1}$

To describe the simple components of $R^{\mathbf{G}}_{\mathbf{L}}\Lambda$, we examine endomorphisms of $R^{\mathbf{G}}_{\mathbf{L}}\Lambda$ for $(\mathbf{L}, \Lambda)$ cuspidal. The ring $\mathcal{H}(\mathbf{L}, \Lambda) = \operatorname{End}(R^{\mathbf{G}}_{\mathbf{L}}\Lambda)$ is called the *Hecke algebra* of $(\mathbf{L}, \Lambda)$. If

$$R^{\mathbf{G}}_{\mathbf{L}}\Lambda = L_1^{m_1} \oplus \cdots \oplus L_k^{m_k}$$

is the decomposition of $R^{\mathbf{G}}_{\mathbf{L}}\Lambda$ into pairwise non-isomorphic irreducibles L_i, then $\mathcal{H}(\mathbf{L}, \Lambda)$ is the product of the matrix algebras Mat_{m_i}, and then $L_1, \ldots, L_k$ are exactly the irreducible modules of $\mathcal{H}(\mathbf{L}, \Lambda)$. Thus, classifying the representations in $\operatorname{Simp}(\mathbf{G}^F \mid (\mathbf{L}, \Lambda))$ is the same as classifying the modules of $\mathcal{H}(\mathbf{L}, \Lambda)$, which is the same as finding idempotents $e_i \in \mathcal{H}(\mathbf{L}, \Lambda)$ corresponding to the projection onto Mat_{m_i}. In particular, the module L_i is $e_i R^{\mathbf{G}}_{\mathbf{L}}(\Lambda)$.

Let's start with the most basic case, $\mathbf{L} = \mathbf{T}$ is a quasi-split torus contained in an F-stable Borel $\mathbf{B}$, and $\Lambda = \mathbb{1}$ is the trivial representation of $\mathbf{T}$. We have $R^{\mathbf{G}}_{\mathbf{T} \subseteq \mathbf{B}}\mathbb{1} = \operatorname{Ind}^{\mathbf{G}^F}_{\mathbf{B}^F}\mathbb{1} \cong \Bbbk G e_{\mathbf{B}^F}$ where $e_{\mathbf{B}^F} = \frac{1}{|B^F|}\sum_{b \in \mathbf{B}^F} b$. This means we have an algebra isomorphism

$$\mathcal{H}(\mathbf{T}, \mathbb{1}) = \operatorname{Hom}(R^{\mathbf{G}}_{\mathbf{T}}\mathbb{1}, R^{\mathbf{G}}_{\mathbf{T}}\mathbb{1}) \cong e_{\mathbf{B}^F}\Bbbk\mathbf{G}^F e_{\mathbf{B}^F}.$$

Recall the Bruhat decomposition: $\mathbf{G}^F$ is the disjoint union of $\mathbf{B}^F w \mathbf{B}^F$ for $w \in \mathbf{W}^F$, where $\mathbf{W} = N_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$ is the Weyl group. Consequently, the elements

$$a_w = \frac{1}{|\mathbf{B}^F|}\sum_{x \in \mathbf{B}^F w \mathbf{B}^F} x \text{ for } w \in \mathbf{W}^F$$

are a basis for $e_{\mathbf{B}^F}\Bbbk\mathbf{G}^F e_{\mathbf{B}^F}$. Let S^F be the set of Coxeter generators for $\mathbf{W}^F$ and $n = |S^F|$.

Theorem (67.2 and 67.4 in [CR])**.** *For $w \in \mathbf{W}^F$ and $s \in S^F$ with $\ell(sw) > \ell(w)$, we have $a_s a_w = a_{sw}$. If $\mathbf{W}^F$ has presentation*

$$\langle s_1, \ldots, s_n \mid s_i^2 = 1,\ \underbrace{s_i s_j s_i \ldots}_{m_{i,j} \text{ terms}} = \underbrace{s_j s_i s_j \ldots}_{m_{i,j} \text{ terms}} \rangle$$

as a Coxeter group, then $\mathcal{H}(\mathbf{T}, \mathbb{1}) \cong e_{\mathbf{B}^F} \Bbbk \mathbf{G}^F e_{\mathbf{B}^F}$ has presentation

$$\langle \Bbbk e, a_1, \ldots, a_n \mid a_i^2 = q_i e + (q_i - 1)a_i,\ \underbrace{a_i a_j a_i \ldots}_{m_{i,j} \text{ terms}} = \underbrace{a_j a_i a_j \ldots}_{m_{i,j} \text{ terms}} \rangle$$

as a $\Bbbk$-algebra, where $e = e_{\mathbf{B}^F}$ is the identity, $a_i = a_{s_i}$ and $q_i = |\mathbf{B}^F|/|s_i \mathbf{B}^F s_i^{-1} \cap \mathbf{B}^F|$.

For an arbitrary Coxeter system (W, S), a commutative ring R and invertible elements $q_i \in R$ for $1 \leq i \leq n$ with $q_i = q_j$ whenever $s_i, s_j \in S$ are conjugate in W, the abstract *Iwahori-Hecke algebra* $\mathcal{H}(W, (q_1, \ldots, q_n))$ is the R-algebra with generators $a_1, \ldots, a_n$ and presentation as in the theorem above. These algebras are very well-studied. It is possible to show, using this algebra and a result called Tits' deformation theorem, that $\mathcal{H}(\mathbf{T}, \mathbb{1})$ is isomorphic to the group algebra $\Bbbk \mathbf{W}^F$ (see [CR, §68] or [DM, §6.2]). Consequently, there is a bijection between the representations in $\mathrm{Simp}(\mathbf{G}^F \mid (\mathbf{T}, \mathbb{1}))$ and the irreducible representations of $\mathbf{W}^F$.

Instead of covering the full correspondence, we'll look at one important example. For a subgroup H of a finite group G, we write $\mathbb{1}_H^G$ for the character of $\mathrm{Ind}_H^G \mathbb{1}$. Recall that for $I \subseteq S^F$ we write $\mathbf{P}_I = \mathbf{B}\mathbf{W}_I^F\mathbf{B}$ for the corresponding parabolic subgroup, and $\mathbf{L}_I$ for the Levi subgroup with $\mathbf{P}_I = \mathbf{L}_I \ltimes R_u(\mathbf{P}_I)$.

Proposition (67.9 in [CR])**.** *The map $\chi \to \hat{\chi}$ from virtual characters of $\mathbf{W}^F$ to virtual characters of $\mathbf{G}^F$ given by*

$$\chi = \sum_{I \subseteq S^F} n_I \mathbb{1}_{\mathbf{W}_J^F}^{\mathbf{W}^F} \ \mapsto\ \hat{\chi} = \sum_{I \subseteq S^F} n_I \mathbb{1}_{\mathbf{P}_I^F}^{\mathbf{G}^F}$$

for $n_I \in \mathbb{Z}$ is well-defined. Moreover, $\langle \chi_1, \chi_2 \rangle_{\mathbf{W}^F} = \langle \hat{\chi}_1, \hat{\chi}_2 \rangle_{\mathbf{G}^F}$ for all χ_1, χ_2.

For a Coxeter system (W, S), it may or may not be possible to write all simple characters in the form $\sum_{I \subseteq S} n_I \mathbb{1}_{W_I}^W$. It is mentioned in [CR] in the remark following 67.9 that it is possible for GL_n (type A_{n-1}), but not for type B_2. However, in every type the character of the *sign representation* of $\Bbbk \mathbf{W}^F$, which is the 1-dimensional representation where the generators s_i act by -1, can be written in this form. The map $\chi \mapsto \hat{\chi}$ then gives us the *Steinberg character*,

$$\mathrm{St}_{\mathbf{G}^F} = \sum_{I \subseteq S^F} (-1)^{|I|} \mathbb{1}_{\mathbf{P}^F}^{\mathbf{G}^F}.$$

This is the character of an irreducible representation appearing in $\mathrm{Simp}(\mathbf{G}^F \mid (\mathbf{T}, \mathbb{1}))$ and not appearing in $\mathrm{Simp}(\mathbf{G}^F \mid (\mathbf{L}_I, \mathbb{1}))$ for any $I \neq \emptyset$ (see [CR, Theorem 67.10]).

Example. Let G be either $\mathrm{SL}_2(\mathbb{F}_q)$ or $\mathrm{GL}_2(\mathbb{F}_q)$ and B the subgroup of upper triangular matrices. If χ is the character of $\Bbbk(G/B)$, then $\mathrm{St}_G = \mathbb{1}_B^G - \mathbb{1}_G^G = \chi - 1$.

Tensoring with the sign representation is an involution on the simple characters of $\mathcal{H}(\mathbf{T}, \mathbb{1})$. This extends to an involution on the simple characters of $\mathbf{G}^F$, and in fact this involution can be described explicitly as

$$D_{\mathbf{G}} = \sum_{I \subseteq S} (-1)^{|I|} R_{\mathbf{L}_I}^{\mathbf{G}} \circ {}^*R_{\mathbf{L}_I}^{\mathbf{G}}$$

(see [DM, §7.2]). This functor is self-adjoint. Moreover, if $(\mathbf{L}_I, \Lambda)$ is a cuspidal pair and γ is the character of a module in $\mathrm{Simp}(\mathbf{G}^F \mid (\mathbf{L}, \Lambda))$, then $(-1)^{|I|} D_{\mathbf{G}} \gamma$ is the character of another module in $\mathrm{Simp}(\mathbf{G}^F \mid (\mathbf{L}, \Lambda))$.

7 Classifying simples of Harish-Chandra series

Now let's go back to the more general case of $\mathcal{H}(\mathbf{L},\Lambda) = \operatorname{End}(R_{\mathbf{L}}^{\mathbf{G}}\Lambda)$ with $(\mathbf{L},\Lambda)$ cuspidal. By adjunction and the Mackey formula we have

$$\begin{aligned}\mathcal{H}(\mathbf{L},\Lambda) &\cong \operatorname{Hom}_{\mathbf{L}^F}(\Lambda, {}^*R_{\mathbf{L}}^{\mathbf{G}} R_{\mathbf{L}}^{\mathbf{G}}\Lambda)\\ &\cong \bigoplus_{x \text{ rep. of } \mathbf{L}^F\backslash \mathcal{S}(\mathbf{L},\mathbf{L})^F/\mathbf{L}^F} \operatorname{Hom}\left(\Lambda, R^{\mathbf{L}}_{\mathbf{L}\cap x\mathbf{L}x^{-1}} {}^*R^{x\mathbf{L}x^{-1}}_{\mathbf{L}\cap x\mathbf{L}x^{-1}}(x\Lambda)\right)\end{aligned}$$

as vector spaces. Since $(\mathbf{L},\Lambda)$ is cuspidal, the restriction on the right is only non-zero when $\mathbf{L} = x\mathbf{L}x^{-1}$, and if $x \in \mathbf{G}$ satisfies this then it is automatically in $S(\mathbf{L},\mathbf{L})^F$ and the double coset $\mathbf{L}x\mathbf{L}$ is equal to the left coset $x\mathbf{L}$. Thus we have

$$\mathcal{H}(\mathbf{L},\Lambda) \;\cong\; \bigoplus_{x \text{ rep. of } N_{\mathbf{G}}(\mathbf{L})^F/\mathbf{L}^F} \operatorname{Hom}(\Lambda, x\Lambda) \;\cong\; \bigoplus_{x\in N_{\mathbf{G}}(\mathbf{L},\Lambda)^F/\mathbf{L}^F} \Bbbk$$

as vector spaces, where $N_{\mathbf{G}^F}(\mathbf{L},\Lambda) = \{x \in \mathbf{G}^F \mid \mathbf{L} = x\mathbf{L}x^{-1} \text{ and } \Lambda \cong x\Lambda\}$. This tells us we should look at the group $\mathbf{W}_{\mathbf{G}}(\mathbf{L},\Lambda) = N_{\mathbf{G}}(\mathbf{L},\Lambda)/\mathbf{L}$, which is called the *relative Weyl group* for the cuspidal pair $(\mathbf{L},\Lambda)$. We can also define a relative Weyl group just for $\mathbf{L}$ by $\mathbf{W}_{\mathbf{G}}(\mathbf{L}) = \mathbf{W}_{\mathbf{G}}(\mathbf{L},\mathbb{1}) = N_{\mathbf{G}}(\mathbf{L})/\mathbf{L}$, and then the usual Weyl group is $\mathbf{W} = \mathbf{W}_{\mathbf{G}}(\mathbf{T}) = \mathbf{W}_{\mathbf{G}}(\mathbf{T},\mathbb{1})$. In summary,

$$\begin{aligned}\mathbf{W}^F &= N_{\mathbf{G}}(\mathbf{T})^F/\mathbf{T}^F &&= \{x\mathbf{T}^F \in \mathbf{G}^F/\mathbf{T}^F \mid x\mathbf{T}x^{-1} = \mathbf{T}\},\\ \mathbf{W}_{\mathbf{G}}(\mathbf{L})^F &= N_{\mathbf{G}}(\mathbf{L})^F/\mathbf{L}^F &&= \{x\mathbf{L}^F \in \mathbf{G}^F/\mathbf{L}^F \mid x\mathbf{L}x^{-1} = \mathbf{L}\},\\ \mathbf{W}_{\mathbf{G}}(\mathbf{L},\Lambda)^F &= N_{\mathbf{G}}(\mathbf{L},\Lambda)^F/\mathbf{L}^F &&= \{x\mathbf{L}^F \in \mathbf{G}^F/\mathbf{L}^F \mid x\mathbf{L}x^{-1} = \mathbf{L} \text{ and } x\Lambda \cong \Lambda\}.\end{aligned}$$

Clearly $\mathbf{W}_{\mathbf{G}}(\mathbf{L},\Lambda)^F$ is a subgroup of $\mathbf{W}_{\mathbf{G}}(\mathbf{L})^F$, and then we can relate both of these back to $\mathbf{W}^F$ as follows.

Proposition (3.4.3 in [DM]). *If $I, J \subseteq S$, the Levi subgroups $\mathbf{L}_I, \mathbf{L}_J$ are conjugate in $\mathbf{G}$ if and only if I, J are conjugate in $\mathbf{W}$ if and only if $\mathbf{W}_I, \mathbf{W}_J$ are conjugate in $\mathbf{W}$. Consequently, we have an isomorphism*

$$\begin{aligned}N_{\mathbf{W}}(\mathbf{W}_I)/\mathbf{W}_I &\cong \mathbf{W}_{\mathbf{G}}(\mathbf{L}_I)\\ (x\mathbf{T})\mathbf{W}_I &\mapsto x\mathbf{L}_I.\end{aligned}$$

For $I \subseteq S$, each coset $w\mathbf{W}_I$ has a unique minimal-length representative (in [DM] these elements are called 'reduced-I').

Lemma (6.1.7 and 6.1.13 in [DM]). *We have $\mathbf{N}_{\mathbf{W}}(\mathbf{W}_I) = \mathbf{N}_I \ltimes \mathbf{W}_I$ where*

$$\mathbf{N}_I = \{\textit{minimal } \mathbf{W}_I\textit{-coset representatives } w \textit{ with } wIw^{-1} = I\}.$$

$\mathbf{N}_I$ is a Coxeter group with generators $\{w_{I\cup\{s\}}w_I \mid s \in S - I\}$, where w_I is the longest element of $\mathbf{W}_I$. Moreover, if I is F-stable then $\mathbf{N}_{\mathbf{W}}(\mathbf{W}_I)^F = \mathbf{N}_I^F \ltimes \mathbf{W}_I^F$. Consequently, we have an isomorphism

$$\begin{aligned}\mathbf{N}_I &\cong \mathbf{W}_{\mathbf{G}}(\mathbf{L}_I)\\ \textit{minimal rep. of } (x\mathbf{T})\mathbf{W}_I &\mapsto x\mathbf{L}_I.\end{aligned}$$

Thus $\mathbf{W}_{\mathbf{G}}(\mathbf{L},\Lambda)^F$ is a subgroup of $\mathbf{W}_{\mathbf{G}}(\mathbf{L})^F$ which can in turn be interpreted as a subgroup of $\mathbf{W}^F$.

For a Coxeter system (W,S), we write $\operatorname{Ref}(W) = \{wsw^{-1} \mid w \in W, s \in S\}$ for the set of reflections in W. If Φ is a root system for W with positive roots Φ^+, then by [DM, Proposition 2.2.10] we have $\operatorname{Ref} W = \{s_\alpha \mid \alpha \in \Phi^+\}$. A *reflection subgroup* of W is a subgroup W' generated by $W' \cap \operatorname{Ref} W$.

Theorem (2.2.11 and 6.2.5 in [DM]). *If (W, S) is a Coxeter system and $W' \subseteq W$ is a reflection subgroup, then $(W', S(W'))$ is a Coxeter system where*

$$S(W') = \{r \in \operatorname{Ref}(w) \mid \{s_\alpha \mid \alpha \in \Phi^+, w(\alpha) < 0\} \cap W' = \{r\}\}.$$

Proposition (6.1.16 in [DM]). *Assume the centre of $\mathbf{G}$ is connected. If $(\mathbf{L}_I, \Lambda)$ is cuspidal, then the group $\mathbf{W}_{\mathbf{G}}(\mathbf{L}_I, \Lambda)^F$ is a reflection subgroup of $\mathbf{W}_{\mathbf{G}}(\mathbf{L}_I)^F \cong \mathbf{N}_I^F$. Consequently, $\mathbf{W}_{\mathbf{G}}(\mathbf{L}_I, \Lambda)^F$ is a Coxeter group, and we denote the generators $S(\mathbf{L}_I, \Lambda)$.*

Theorem (6.2.6 in [DM] or 3.1.28 in [GM] (more general)). *For $(\mathbf{L}, \Lambda)$ cuspidal, we have an isomorphism*

$$\operatorname{End}_{\mathbf{G}^F}(R_{\mathbf{L}}^{\mathbf{G}} \Lambda) \cong \mathcal{H}(\mathbf{W}_{\mathbf{G}}(\mathbf{L}, \Lambda)^F, (q^{c_s})_{s \in S(\mathbf{L}, \Lambda)})$$

for some integers c_s. Consequently, there is a bijection between $\operatorname{Simp}(\mathbf{G}^F | (\mathbf{L}, \Lambda))$ and the simple characters of $\mathbf{W}_{\mathbf{G}}(\mathbf{L}, \Lambda)^F$.

8 Notes on references

I used the references [DM], [GM] and [CR] for these notes, which each take a very different approach to describing Harish-Chandra theory. Here are some comments on each in case people want to look further into these references:

- [CR] only develops the theory for $R_{\mathbf{T}}^{\mathbf{G}} \mathbb{1}$ as far as I can tell, but gives a very intuitive and explicit description of this case.
- [DM] spends a lot of time describing the isomorphism between $\operatorname{End}_{\mathbf{G}^F}(R_{\mathbf{L}}^{\mathbf{G}} \Lambda)$ and an Iwahori-Hecke algebra in section 6. They are pretty explicit, but leave some details to other references (like the integers c_s in Theorem 6.2.6). Writing out the isomorphism explicitly would mean tracing a whole bunch of definitions throughout the textbook and several research papers, which I am not interested in doing.
- [GM] devotes section 2 to the more general form of $R_{\mathbf{L}}^{\mathbf{G}}$ that we will see in later talks, and only goes into Harish-Chandra theory in section 3. They opt for an excruciating level of generality by studying the theory for any finite group with a BN-pair, which requires a 'twisted extended Iwahori-Hecke algebra'.

References

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